PART A

Unit - I

Differential Equations - 1

1.1 Introduction

We are familiar with the solution of differential equations (d.e.) of first order and first degree. In this unit we discuss the solution of differential equations of first order but not of first degree. In addition to the general solution and particular solution associated with the d.e, we also introduce singular solution. The d.es of first order but not of first degree are also branded as $p - y - x$ equations.

1.2 Differential equations of first order and higher degree

If $y = f(x)$, we use the notation $\dfrac{dy}{dx} = p$ throughout this unit.

A differential equation of first order and n^{th} degree is the form

$$A_0 p^n + A_1 p^{n-1} + A_2 p^{n-2} + \cdots + A_n = 0 \qquad \ldots (1)$$

where $A_0, A_1, A_2, \ldots A_n$ are functions of x and y. This being a d.e of first order, the associated general solution will contain only one arbitrary constant. We proceed to discuss equations solvable for p or y or x, wherein the problem is reduced to that of solving one or more differential equations of first order and first degree. We finally disucss the solution of Clairaut's equation.

1.21 Equations solvable for p

Supposing that the LHS of (1) is expressed as a product of n linear factors, then the equivalent form of (1) is

$$[p - f_1(x, y)]\,[p - f_2(x, y)] \ldots [p - f_n(x, y)] = 0 \qquad \ldots (2)$$

$$\Rightarrow \quad [p - f_1(x, y)] = 0, \quad [p - f_2(x, y)] = 0, \ldots [p - f_n(\ , y)] = 0$$

All these are differential equations of first order and first degree. They can be solved by the known methods. If $F_1(x, y, c) = 0, F_2(x, y, c) = 0, \ldots F_n(x, y, c) = 0$ respectively represents the solution of these equations then the **general solution is given by the product of all these solutions.**

Note : *We need to present the general solution with the same arbitrary constant in each factor.*

WORKED PROBLEMS

1. *Solve* : $\left(\frac{dy}{dx}\right)^2 - 7\left(\frac{dy}{dx}\right) + 12 = 0$

>> The given equation is

$$p^2 - 7p + 12 = 0$$

ie., $(p-3)(p-4) = 0$ or $p = 3, 4$

We have,

$$\frac{dy}{dx} = 3 \Rightarrow y = 3x + c \quad \text{or} \quad y - 3x - c = 0$$

Also, $\frac{dy}{dx} = 4 \Rightarrow y = 4x + c$ or $y - 4x - c = 0$

Thus the general solution is given by, $\mathbf{(y - 3x - c)(y - 4x - c) = 0}$

2. Solve : $y\left(\frac{dy}{dx}\right)^2 + (x-y)\frac{dy}{dx} - x = 0$

>> The given equation is

$$y p^2 + (x-y) p - x = 0$$

$$\therefore \quad p = \frac{-(x-y) \pm \sqrt{(x-y)^2 + 4xy}}{2y}$$

$$p = \frac{(y-x) \pm (x+y)}{2y}$$

ie., $p = \frac{y - x + x + y}{2y}$ or $p = \frac{y - x - x - y}{2y}$

ie., $p = 1$ or $p = -x/y$

We have,

$$\frac{dy}{dx} = 1 \Rightarrow y = x + c \quad \text{or} \quad (y - x - c) = 0$$

Also, $\frac{dy}{dx} = \frac{-x}{y}$ or $y\,dy + x\,dx = 0 \Rightarrow \int y\,dy + \int x\,dx = k$

ie., $\frac{y^2}{2} + \frac{x^2}{2} = k$ or $y^2 + x^2 = \underline{\underline{2k}}$ or $(x^2 + y^2 - c) = 0$

Thus the general solution is given by $\mathbf{(y - x - c)(x^2 + y^2 - c) = 0}$

3. *Solve* : $xy\left(\frac{dy}{dx}\right)^2 - (x^2+y^2)\frac{dy}{dx} + xy = 0$

>> The given equation is

$$xy\,p^2 - (x^2+y^2)\,p + xy = 0$$

$$\therefore \quad p = \frac{(x^2+y^2) + \sqrt{(x^2+y^2)^2 - 4x^2y^2}}{2xy}$$

$$p = \frac{(x^2+y^2) \pm (x^2-y^2)}{2xy}$$

ie., $p = \frac{x^2+y^2+x^2-y^2}{2xy}$ or $p = \frac{x^2+y^2-x^2+y^2}{2xy}$

ie., $p = \frac{x}{y}$ or $p = \frac{y}{x}$

We have,

$$\frac{dy}{dx} = \frac{x}{y} \quad \text{or} \quad y\,dy - x\,dx = 0 \quad \Rightarrow \quad \int y\,dy - \int x\,dx = k$$

ie., $\frac{y^2}{2} - \frac{x^2}{2} = k$ or $y^2 - x^2 = 2k$ or $(y^2 - x^2 - c) = 0$

Also, $\frac{dy}{dx} = \frac{y}{x}$ or $\frac{dy}{y} - \frac{dx}{x} = 0 \Rightarrow \int\frac{dy}{y} - \int\frac{dx}{x} = k$

ie., $\log y - \log x = k$ or $\log(y/x) = \log c \Rightarrow y - cx = 0$

Thus the general solution is given by $(y^2 - x^2 - c)\,(y - cx) = 0$

---

4. *Solve* : $p^2 - 2p\sin hx - 1 = 0$

>> $p^2 - 2p\sin hx - 1 = 0$, by data.

$$\therefore \quad p = \frac{2\sin h\,x \pm \sqrt{4\sin h^2 x + 4}}{2}$$

$$p = \frac{2\sin hx \pm 2\sqrt{\cosh^2 x}}{2} = \sin h\,x \pm \cos h\,x$$

ie., $p = \sin h\,x + \cos h\,x$ or $p = \sin h\,x - \cos h\,x$

We have,

$$\frac{dy}{dx} = \sin h\,x + \cos h\,x \quad \Rightarrow \quad y = \cos h\,x + \sin h\,x + c$$

Also, $\dfrac{dy}{dx} = \sin h\, x - \cos h\, x \quad \Rightarrow \quad y = \cos h\, x - \sin h\, x + c$

Thus the general solution is given by

$$(y - \cos h\, x - \sin h\, x - c)\ (y - \cos h\, x + \sin h\, x - c) = 0$$

or

$$\left(y - \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} - c\right)\left(y - \frac{e^x + e^{-x}}{2} + \frac{e^x - e^{-x}}{2} - c\right) = 0$$

ie., $(y - e^x - c)\ (y - e^{-x} - c) = 0$

---

5. *Solve :* $p^2 + 2py \cot x = y^2$

>> $p^2 + 2py \cot x - y^2 = 0$, by data.

$$\therefore \quad p = \frac{-2y \cot x \pm \sqrt{4y^2 \cot^2 x + 4y^2}}{2}$$

$$p = \frac{-2y \cot x \pm 2y \operatorname{cosec} x}{2} = y(-\cot x \pm \operatorname{cosec} x)$$

ie., $p = y(-\cot x + \operatorname{cosec} x)$ or $p = y(-\cot x - \operatorname{cosec} x)$

We have,

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$$

or $\dfrac{dy}{y} = (\operatorname{cosec} x - \cot x)\, dx$

$\Rightarrow \displaystyle\int \frac{dy}{y} = \int (\operatorname{cosec} x - \cot x)\, dx + k$

ie., $\log y = \log \tan(x/2) - \log \sin x + k$

or $\log y = \log\left[\dfrac{c \tan(x/2)}{\sin x}\right]$ where $\log c = k$

$\Rightarrow \quad y = \dfrac{c \tan(x/2)}{\sin x} = \dfrac{c \tan(x/2)}{2 \sin(x/2) \cos(x/2)} = \dfrac{c}{2 \cos^2 x/2} = \dfrac{c}{1 + \cos x}$

ie., $y(1 + \cos x) = c$

Also $\dfrac{dy}{dx} = y(-\cot x - \operatorname{cosec} x) = -y(\cot x + \operatorname{cosec} x)$

or $\dfrac{dy}{-y} = (\cot x + \operatorname{cosec} x)\, dx$

$$\int \frac{dy}{y} + \int (\cot x + \operatorname{cosec} x)\, dx = k$$

ie., $\log y + \log \sin x + \log \tan (x/2) = k$

or $\log [y \sin x \tan (x/2)] = \log c$ where $\log c = k$

$\Rightarrow$ $y \cdot 2 \sin (x/2) \cos (x/2) \dfrac{\sin (x/2)}{\cos (x/2)} = c$

ie., $y \cdot 2 \sin^2 (x/2) = c$ or $y (1 - \cos x) = c$

Thus the general solution is $\mathbf{[y (1 + \cos x) - c] [y (1 - \cos x) - c] = 0}$

---

6. *Solve :* $x (y')^2 - (2x + 3y) y' + 6y = 0$

>> The given equation with the usual notation is,

$$x p^2 - (2x + 3y)p + 6y = 0$$

$\therefore$ $p = \dfrac{(2x + 3y) \pm \sqrt{(2x + 3y)^2 - 24xy}}{2x}$

$$p = \frac{(2x + 3y) \pm (2x - 3y)}{2x} = 2 \quad \text{or} \quad \frac{3y}{x}$$

We have,

$$\frac{dy}{dx} = 2 \Rightarrow \int dy = 2 \int dx + c \quad \text{or} \quad y = 2x + c \quad \text{or} \quad (y - 2x - c) = 0$$

Also $\dfrac{dy}{dx} = \dfrac{3y}{x}$ or $\dfrac{dy}{y} = 3 \dfrac{dx}{x}$ > $\displaystyle\int \frac{dy}{y} = 3 \int \frac{dx}{x} + k$

ie., $\log y = 3 \log x + k$ or $\log y = \log x^3 + \log c$, where $k = \log c$

ie., $\log y = \log (cx^3)$ $\Rightarrow$ $y = cx^3$ or $y - cx^3 = 0$

Thus the general solution is $\mathbf{(y - 2x - c) (y - cx^3) = 0}$

---

7. *Solve :* $x^2 p^2 + xp - (y^2 + y) = 0$

>> $x^2 p^2 + xp - (y^2 + y) = 0$, by data.

$\therefore$ $p = \dfrac{-x \pm \sqrt{x^2 + 4x^2 (y^2 + y)}}{2x^2}$

$$= \frac{-x \pm \sqrt{x^2 + 4x^2 y^2 + 4x^2 y}}{2x^2} = \frac{-x \pm \sqrt{(x + 2xy)^2}}{2x^2}$$

$$p = \frac{-x+x+2xy}{2x^2} \quad \text{or} \quad p = \frac{-x-x-2xy}{2x^2} = \frac{-2x-2xy}{2x^2}$$

ie., $$p = \frac{y}{x} \quad \text{or} \quad p = \frac{-1}{x} - \frac{y}{x}$$

We have,

$$\frac{dy}{dx} = \frac{y}{x} \quad \text{or} \quad \frac{dy}{y} = \frac{dx}{x} \quad \Rightarrow \quad \int \frac{dy}{y} = \int \frac{dx}{x} + k$$

ie., $\log y = \log x + \log c$, where $k = \log c$

ie., $\log y = \log(cx) \Rightarrow y - cx = 0$

Also, $$\frac{dy}{dx} + \frac{y}{x} = \frac{-1}{x}$$

This is a linear d.e of the form $\dfrac{dy}{dx} + Py = Q$ whose solution is,

$$y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + c, \quad \text{where } P = \frac{1}{x}, \ Q = \frac{-1}{x}$$

Now, $e^{\int P\,dx} = e^{\int 1/x} = e^{\log x} = x$

Hence, $y \cdot x = \int -\dfrac{1}{x} \cdot x\,dx + c$ or $xy = -x + c$ or $xy + x - c = 0$

Thus the general solution is $(y - cx)\ (xy + x - c) = 0$.

- -

⑧ Solve : $p(p+y) = x(x+y)$

>> The given equation is, $p^2 + py - x(x+y) = 0$

$$\therefore \quad p = \frac{-y \pm \sqrt{y^2 + 4x(x+y)}}{2}$$

$$p = \frac{-y \pm \sqrt{4x^2 + 4xy + y^2}}{2} = \frac{-y \pm (2x+y)}{2}$$

ie., $$p = x \quad \text{or} \quad p = \frac{-2(y+x)}{2} = -(y+x)$$

We have,

$$\frac{dy}{dx} = x \Rightarrow y = \frac{x^2}{2} + k \quad \text{or} \quad 2y - x^2 - c = 0, \quad \text{where } c = 2k$$

Also, $$\frac{dy}{dx} = -(y+x)$$

ie., $\frac{dy}{dx} + y = -x$, is a linear d.e (*Similar to the previous problem*)

$$P = 1, \quad Q = -x \; ; \quad e^{\int P\,dx} = e^x$$

Hence $y\,e^x = \int -x\,e^x\,dx + c$

ie., $y\,e^x = -(\,x\,e^x - e^x\,) + c$, integrating by parts.

or $e^x\,(\,y + x - 1\,) - c = 0$

Thus the general solution is given by $\mathbf{(\,2y - x^2 - c\,)\,[\,e^x\,(\,y + x - 1\,) - c\,] = 0}$

---

9. *Solve :* $4y^2p^2 + 2pxy\,(\,3x + 1\,) + 3x^3 = 0$

>> $4y^2p^2 + 2xy\,(\,3x + 1\,)\,p + 3x^3 = 0$, by data.

$$\therefore \quad p = \frac{-2xy\,(\,3x+1\,) \pm \sqrt{4x^2y^2\,(\,3x+1\,)^2 - 48\,x^3\,y^2}}{8y^2}$$

$$= \frac{-6x^2\,y - 2xy \pm \sqrt{36\,x^4\,y^2 + 24\,x^3\,y^2 + 4x^2\,y^2 - 48\,x^3\,y^2}}{8y^2}$$

$$= \frac{-6x^2\,y - 2xy \pm \sqrt{36\,x^4\,y^2 - 24\,x^3\,y^2 + 4x^2\,y^2}}{8y^2}$$

$$= \frac{-6x^2\,y - 2xy \pm (\,6x^2\,y - 2xy\,)}{8y^2}$$

$$p = \frac{-4xy}{8y^2} = \frac{-x}{2y} \quad \text{or} \quad p = \frac{-12\,x^2\,y}{8y^2} = \frac{-3x^2}{2y}$$

We have,

$$\frac{dy}{dx} = \frac{-x}{2y} \quad \text{or} \quad x\,dx + 2y\,dy = 0 \quad \Rightarrow \quad \int x\,dx + \int 2y\,dy = k$$

ie., $\frac{x^2}{2} + y^2 = k$ or $x^2 + 2y^2 - c = 0$, where $2k = c$

Also, $\frac{dy}{dx} = \frac{-3x^2}{2y}$ or $3x^2\,dx + 2y\,dy = 0 \Rightarrow \int 3x^2\,dx + \int 2y\,dy = c$

ie., $x^3 + y^2 = c$ or $x^3 + y^2 - c = 0$

Thus the general solution is $\mathbf{(\,x^2 + 2y^2 - c\,)\,(\,x^3 + y^2 - c\,) = 0}$

---

10. *Solve* : $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

>> With the notation $p = \frac{dy}{dx}$, we have $\frac{1}{p} = \frac{dx}{dy}$

The given equation is,

$$p - \frac{1}{p} = \frac{x}{y} - \frac{y}{x} \quad \text{or} \quad \frac{p^2 - 1}{p} = \frac{x}{y} - \frac{y}{x}$$

ie., $$p^2 - p\left(\frac{x}{y} - \frac{y}{x}\right) - 1 = 0$$

$$\therefore \quad p = \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \sqrt{\left(\frac{x}{y} - \frac{y}{x}\right)^2 + 4}}{2}$$

$$p = \frac{\left(\frac{x}{y} - \frac{y}{x}\right) \pm \left(\frac{x}{y} + \frac{y}{x}\right)}{2} \; ; \; p = \frac{x}{y} \quad \text{or} \quad \frac{-y}{x}$$

We have, $\frac{dy}{dx} = \frac{x}{y}$ or $x\,dx - y\,dy = 0 \Rightarrow \int x\,dx - \int y\,dy = k$

ie., $\frac{x^2}{2} - \frac{y^2}{2} = k$ or $x^2 - y^2 - c = 0$, where $c = 2k$

Also, $\frac{dy}{dx} = \frac{-y}{x}$ or $\frac{dy}{y} = \frac{-dx}{x} \Rightarrow \int \frac{dy}{y} + \int \frac{dx}{x} = k$

ie., $\log y + \log x = k$ or $\log(xy) = \log c \Rightarrow xy - c = 0$

Thus the general solution is $(x^2 - y^2 - c)(xy - c) = 0$

1.22 Equations solvable for y

We say that the given differential equation is ***solvable for y***, if it is possible to express y in terms of x and p explicitly. The method of solving is illustrated stepwise.

- $y = f(x, p)$ . . .(1)
- We differentiate (1) w.r.t x to obtain

$$\frac{dy}{dx} = p = F\left(x, y, \frac{dp}{dx}\right) \qquad \ldots(2)$$

Here it should be noted that there is no need to have the given equation solvable for y in the explicit form (1).

By recognizing that the equation is solvable for y, we can proceed to differentiate the same w.r.t x.

- We notice that (2) is a d.e of first order in p and x. We solve the same to obtain the solution in the form.

$$\phi(x, p, c) = 0 \qquad \ldots(3)$$

- By eliminating p from (1) and (3) we obtain the general solution of the given d.e in the form $G(x, y, c) = 0$.

Remark : *Suppose we are unable to eliminate p from (1) and (3), we need to solve for x and y from the same to obtain.*

$$x = F_1(p, c), \quad y = F_2(p, c)$$

which constitutes the solution of the given equation regarding p as a parameter.

1.23 Equations solvable for x

We say that the given equation is ***solvable for x,*** if it is possible to express x in terms of y and p. The method of solving is identical with that of the earlier one and the same is as follows.

- $x = f(y, p)$ $\ldots(1)$
- Differentiate w.r.t. y to obtain

$$\frac{dx}{dy} = \frac{1}{p} = F\left(x, y, \frac{dp}{dy}\right) \qquad \ldots(2)$$

- (2) being a d.e of first order in p and y the solution is of the form.

$$\phi(y, p, c) = 0 \qquad \ldots(3)$$

- By eliminating p from (1) and (3) we obtain the general solution of the given d.e in the form $G(x, y, c) = 0$

Note : *The content of the remark given in the previous article continue to hold good here also.*

1.24 Singular Solution

Let us suppose that the general solution of the given d.e is

$$G(x, y, c) = 0 \qquad \ldots(1)$$

Treating c as a parameter we differentiate (1) partially w.r.t c to obtain

$$\frac{\partial}{\partial c}\,[G(x, y, c) = 0 \qquad \ldots(2$$

Eliminating the parameter c from (1) and (2) we obtain a relation of the form $\phi(x, y) = 0$ and is known as the ***Singular solution*** of the given d.e.

1.25 Geometrical significance of the singular solution

Let $f(x, y, c) = 0$ be the general solution of a differential equation of first order. For each value of the arbitrary constant c, $f(x, y, c) = 0$ represents a curve and hence $f(x, y, c) = 0$ represents a family of curves.

The ***Envelope*** of a family of curves is a curve which touches each member of the family.

If the family of curves $f(x, y, c) = 0$ possess an envelope, the equation of the envelope is the singular solution of the d.e

WORKED PROBLEMS

11 *Solve* : $x p^2 + x = 2yp$

>> We present the solution of this equation in all the three methods. That is, solvable for p, solvable for y and solvable for x.

Method-1 : We explore the option of **solving for p**. We consider the given equation in the form.

$$x p^2 - 2yp + x = 0$$

$$\therefore \quad p = \frac{2y \pm \sqrt{4y^2 - 4x^2}}{2x} = \frac{y \pm \sqrt{y^2 - x^2}}{x}$$

We have, $\dfrac{dy}{dx} = \dfrac{y + \sqrt{y^2 - x^2}}{x}$

This being a homogeneous d.e, we solve by putting $y = vx$. This gives,

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ and the equation becomes

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{v^2 x^2 - x^2}}{x} = v + \sqrt{v^2 - 1}$$

ie., $$x\frac{dv}{dx} = \sqrt{v^2 - 1}$$

or $$\frac{dv}{\sqrt{v^2 - 1}} = \frac{dx}{x} \quad \Rightarrow \quad \int \frac{dv}{\sqrt{v^2 - 1}} - \int \frac{dx}{x} = k$$

ie., $$\log\left(v + \sqrt{v^2 - 1}\right) - \log x = k \quad \text{or} \quad \log\left(\frac{v + \sqrt{v^2 - 1}}{x}\right) = \log c$$

$$\Rightarrow \quad v + \sqrt{v^2 - 1} = cx, \quad \text{where } v = y/x$$

ie., $$\frac{y}{x} + \frac{\sqrt{y^2 - x^2}}{x} = cx, \quad \text{or} \quad y + \sqrt{y^2 - x^2} = cx^2$$

Also if $\frac{dy}{dx} = \frac{y - \sqrt{y^2 - x^2}}{x}$, we obtain $x\,\frac{dv}{dx} = -\sqrt{v^2 - 1}$

$$\Rightarrow \quad \int \frac{dv}{\sqrt{v^2 - 1}} + \int \frac{dx}{x} = k$$

ie., $\log(v + \sqrt{v^2 - 1}) + \log x = k$

or $\log[(v + \sqrt{v^2 - 1})x] = \log c$

$\Rightarrow$ $(v + \sqrt{v^2 - 1})x = c$, where $v = y/x$

ie., $\left[\frac{y}{x} + \frac{\sqrt{y^2 - x^2}}{x}\right]x = c$ or $y + \sqrt{y^2 - x^2} = c$

Thus the general solution is given by

$$(y + \sqrt{y^2 - x^2} - cx^2)\,(y + \sqrt{y^2 - x^2} - c) = 0$$

Method-2 We explore the option of **solving for y**

$$x p^2 + x = 2yp \qquad \ldots(1)$$

That is, $y = \frac{x(p^2 + 1)}{2p}$ which is of the form $y = f(x, p)$. We conclude that **the equation is solvable for y**. We must prefer to use (1) itself for differentiating w.r.t x. Hence we have,

$$x \cdot 2p\frac{dp}{dx} + 1 \cdot p^2 + 1 = 2\left(y\frac{dp}{dx} + \frac{dy}{dx} \cdot p\right)$$

ie., $2xp\frac{dp}{dx} + p^2 + 1 = 2y\frac{dp}{dx} + 2p^2$, since $\frac{dy}{dx} = p$

ie., $2\frac{dp}{dx}(xp - y) = p^2 - 1$

ie., $2\frac{dp}{dx}\left[xp - \frac{x(p^2 + 1)}{2p}\right] = p^2 - 1$

ie., $2x\frac{dp}{dx}\,\frac{(p^2 - 1)}{2p} = (p^2 - 1)$

ie., $\frac{x}{p}\frac{dp}{dx} = 1$ or $\frac{dp}{p} = \frac{dx}{x}$ $\Rightarrow$ $\int \frac{dp}{p} = \int \frac{dx}{x} + k$

ie., $\log p - \log x = k$ or $\log(p/x) = \log c$ $\rightarrow$ $p = cx$

Now, we need to eliminate p from the relations,

$$x\,p^2 + x = 2yp \qquad \ldots (1)$$

and $$p = cx \qquad \ldots (2)$$

Using (2) in (1) we have,

$$x \cdot c^2 x^2 + x = 2y \cdot cx \quad \text{or} \quad c^2 x^2 + 1 = 2yc$$

Thus $\mathbf{c^2 x^2 + 1 = 2yc}$ is the general solution.

Method - 3 we explore the option of solving for x.

$$x\,p^2 + x = 2yp$$

That is $x\,(p^2 + 1) = 2yp$ or $x = \dfrac{2yp}{p^2+1}$ which is of the form $x = f(y, p)$

We conclude that the equation is **solvable for x**. We use (1) itself to differentiate w.r.t y. Hence we have,

$$x \cdot 2p\,\frac{dp}{dy} + \frac{dx}{dy} \cdot p^2 + \frac{dx}{dy} = 2\left(y\,\frac{dp}{dy} + 1.p \right)$$

ie., $$2xp\,\frac{dp}{dy} + \frac{1}{p} \cdot p^2 + \frac{1}{p} = 2y\,\frac{dp}{dy} + 2p \quad \left(\textit{Since } p = \frac{dy}{dx},\ \textit{we have } \frac{1}{p} = \frac{dx}{dy} \right)$$

ie., $$2xp\,\frac{dp}{dy} + p + \frac{1}{p} = 2y\,\frac{dp}{dy} + 2p$$

ie., $$2\,\frac{dp}{dy}\,(xp - y) = p - \frac{1}{p}$$

ie., $$2\,\frac{dp}{dy}\left[\frac{2yp}{p^2+1} \cdot p - y \right] = \frac{p^2-1}{p}$$

ie., $$2y\,\frac{dp}{dy}\left[\frac{p^2-1}{p^2+1} \right] = \frac{p^2-1}{p}$$

ie., $$\frac{2y}{p^2+1}\,\frac{dp}{dy} = \frac{1}{p}$$

or $$\frac{2p}{p^2+1}\,dp = \frac{dy}{y} \Rightarrow \int \frac{2p}{p^2+1}\,dp - \int \frac{dy}{y} = k$$

ie., $$\log\left(\frac{p^2+1}{y} \right) = \log c \Rightarrow p^2 + 1 = cy$$

Now, we need to eliminte p from the relations

$$x p^2 + x = 2yp \qquad \ldots (1)$$

and $\quad p^2 + 1 = cy \qquad \ldots (2)$

From (2), $p^2 = cy - 1$ and $p = \sqrt{cy-1}$

Hence (1) becomes,

$$x\,(cy-1) + x = 2y\sqrt{cy-1}$$

or $\quad cxy = 2y\sqrt{cy-1}$ or $cx = 2\sqrt{cy-1}$

Equivalently, $c^2 x^2 = 4(cy-1)$ or $c^2 x^2 + 4 = 4cy$

Thus $\mathbf{c^2 x^2 + 4 = 4cy}$ is the general solution.

Remark : *Observe the solution obtained by the three methods.*

1. *Solvable for p :* $y + \sqrt{y^2 - x^2} = cx^2$ *and* $y + \sqrt{y^2 - x^2} = c$
2. *Solvable for y :* $c^2 x^2 + 1 = 2cy$
3. *Solvable for x :* $c^2 x^2 + 4 = 4cy$

The solution structure in (2) and (3) are the same. Also in the case (1) we have,

$$(y - cx^2)^2 = y^2 - x^2 \quad \textit{and} \quad (y-c)^2 = y^2 - x^2$$

ie., $\quad y^2 - 2cy\,x^2 + c^2 x^4 = y^2 - x^2$ *and* $y^2 - 2cy + c^2 = y^2 - x^2$

or $\quad -2cy + c^2 x^2 = -1$ *and* $c^2 + x^2 = 2cy$

ie., $\quad c^2 x^2 + 1 = 2cy$ *and* $x^2 + c^2 = 2cy$

It is also evident that the solution structure in the case (1) is same compared to the cases (2) and (3).

Note : ***Singular solution of the given equation.***

We consider the general solution,

$$c^2 x^2 + 1 = 2cy \qquad \ldots (1)$$

We differentiate partially w.r.t. c, treating c as a parameter. That is,

$$2cx^2 = 2y \quad or \quad c = y/x^2 \qquad \ldots (2)$$

We need to eliminate c from (1) and (2).

Using (2) in (1) we have,

$$\frac{y^2}{x^4}\cdot x^2 + 1 = 2\cdot\frac{y}{x^2}\cdot y \quad or \quad \frac{y^2}{x^2} + 1 = \frac{2y^2}{x^2}$$

or $y^2 + x^2 = 2y^2$ *or* $y^2 = x^2 \Rightarrow y = \pm x$

$y = \pm x$ constitutes the singular solution of the given equation. Geometrically, we can say that $y = \pm x$, the envelope, being a straight line passing through the origin, touches every member of the family of curves represented by $c^2 x^2 + 1 = 2cy$

12. *Solve* : $y - 2px = \tan^{-1}(xp^2)$

>> By data, $y = 2px + \tan^{-1}(xp^2)$ ... (1)

The equation is of the form $y = f(x, p)$, solvable for y.

Differentiating (1) w.r.t. x,

$$p - 2p - 2\frac{dp}{dx}x = \frac{1}{1 + x^2p^4}\left[x \cdot 2p\frac{dp}{dx} + p^2\right]$$

ie., $$-p - 2x\frac{dp}{dx} = \frac{1}{1 + x^2p^4}\left[2xp\frac{dp}{dx} + p^2\right]$$

ie., $$-p - \frac{p^2}{1 + x^2p^4} = 2x\frac{dp}{dx}\left[\frac{p}{1 + x^2p^4} + 1\right]$$

ie., $$-p\left[\frac{1 + x^2p^4 + p}{1 + x^2p^4}\right] = 2x\frac{dp}{dx}\left[\frac{p + 1 + x^2p^4}{1 + x^2p^4}\right]$$

or $-p = 2x\frac{dp}{dx}$ or $\frac{dx}{x} = -2\frac{dp}{p} \Rightarrow \int\frac{dx}{x} + 2\int\frac{dp}{p} = k$

ie., $\log x + 2\log p = k$ or $\log(xp^2) = \log c \Rightarrow xp^2 = c$

Consider $y = 2px + \tan^{-1}(xp^2)$ ... (1)

and $xp^2 = c$ or $p = \sqrt{c/x}$ ... (2)

Using (2) in (1) we have,

$$y = 2\sqrt{c/x} \cdot x + \tan^{-1}(c)$$

Thus $\boldsymbol{y = 2\sqrt{cx} + \tan^{-1} c}$, is the general solution.

13. *Obtain the general solution and the singular solution of the equation* $y + px = p^2 x^4$

>> The given equation is solvable for y only.

$$y + px = p^2 x^4 \qquad \ldots (1)$$

Differentiating w.r.t x,

$$p + p + x\frac{dp}{dx} = 4p^2 x^3 + 2px^4 \frac{dp}{dx}$$

ie., $$2p - 4p^2 x^3 = x\frac{dp}{dx}\,(2px^3 - 1).$$

ie., $$-2p\,(2px^3 - 1) = x\frac{dp}{dx}\,(2px^3 - 1)$$

ie., $$-2p = x\frac{dp}{dx} \text{ or } \frac{dx}{x} = \frac{-dp}{2p} \Rightarrow \int \frac{dx}{x} + \frac{1}{2}\int \frac{dp}{p} = k$$

ie., $$\log x + \log\sqrt{p} = k \text{ or } \log(x\sqrt{p}) = \log c \Rightarrow x\sqrt{p} = c$$

Consider, $y + px = p^2 x^4 \qquad \ldots (1)$

$$x\sqrt{p} = c \quad \text{or} \quad x^2 p = c \quad \text{or} \quad p = c/x^2 \qquad \ldots (2)$$

Using (2) in (1) we have, $y + (c/x^2)\,x = (c^2/x^4)\,x^4$ or $y + (c/x) = c^2$

Thus $xy + c = c^2 x$ **is the general solution.**

Now, to obtain the singular solution, we differentiate this relation partially w.r.t c, treating c as a parameter.

That is, $1 = 2cx$ or $c = 1/2x$.

The general solution now becomes,

$$xy + \frac{1}{2x} = \frac{1}{4x^2}x \text{ or } xy = -\frac{1}{4x}$$

Thus $4x^2 y + 1 = 0$, **is the singular solution.**

---

14. *Solve,* $x^2 p^4 + 2xp - y = 0$ *by obtaining the general and singular solution.*

>> The given equation is solvable for y only.

$$x^2 p^4 + 2xp - y = 0 \qquad \ldots (1)$$

Differentiating w.r.t x,

$$4x^2 p^3 \frac{dp}{dx} + 2xp^4 + 2x\frac{dp}{dx} + 2p - p = 0$$

ie., $$2x\frac{dp}{dx}\,[\,2xp^3+1\,]+p\,[\,2x\,p^3+1\,]=0$$

or $$2x\frac{dp}{dx}+p=0 \quad \text{or} \quad \frac{dp}{p}=\frac{-dx}{2x} \quad \Rightarrow \quad \int\frac{dp}{p}+\frac{1}{2}\int\frac{dx}{x}=k$$

ie., $$\log(p\sqrt{x})=\log c \quad \Rightarrow \quad p\sqrt{x}=c$$

Consider, $x^2p^4+2xp-y=0$ . . . (1)

and $p\sqrt{x}=c$ or $p=c/\sqrt{x}$ . . . (2)

Using (2) and (1) we have,

$$x^2\cdot\frac{c^4}{x^2}+2x\cdot\frac{c}{\sqrt{x}}-y=0$$

ie., $c^4+2c\sqrt{x}=y$ is the general solution.

Thus by setting $c^2=k$, the general solution assumes the form

$$\boldsymbol{k^2+2\sqrt{kx}=y}$$

Now differentiating partially w.r.t. k we have,

$$2k+2\sqrt{x}\,\frac{1}{2\sqrt{k}}=0$$

ie., $$k+\sqrt{x/k}=0 \quad \text{or} \quad k=-\sqrt{x/k} \quad \text{or} \quad k^2=x/k \quad \text{or} \quad k^3=x$$

Using $k=x^{1/3}$ in the general solution we have,

$$(x^{1/3})^2+2\sqrt{x}\,(x^{1/3})^{1/2}=y$$

ie., $$x^{2/3}+2\,x^{2/3}=y \quad \text{or} \quad 3x^{2/3}=y \quad \Rightarrow \quad 27x^2=y^3$$

Thus $y^3=27x^2$ is the singular solution.

18. Solve : $y=p\sin p+\cos p$

\>> $y=p\sin p+\cos p$ . . . (1)

Differentiating w.r.t. x,

$$p=p\cos p\frac{dp}{dx}+\sin p\frac{dp}{dx}-\sin p\frac{dp}{dx}$$

ie., $$1=\cos p\frac{dp}{dx} \quad \text{or} \quad \cos p\,dp=dx \quad \Rightarrow \quad \int\cos p\,dp=\int dx+c$$

ie., $$\sin p=x+c \quad \text{or} \quad x=\sin p-c$$

Thus we can say that $\boldsymbol{y=p\sin p+\cos p}$ and $\boldsymbol{x=\sin p-c}$ constitutes the general solution of the given d.e

Note : $\sin p = x + c \Rightarrow p = \sin^{-1}(x+c)$.

We can as well substitute for p in (1) and present the solution in the form,

$$y = (x+c)\sin^{-1}(x+c) + \cos\sin^{-1}(x+c)$$

16. *Solve :* $y = 3x + \log p$

>> $y = 3x + \log p$ $\quad\ldots(1)$

Differentiating w.r.t x,

$$p = 3 + \frac{1}{p}\frac{dp}{dx}$$

or $\quad (p-3)p = \dfrac{dp}{dx}$ $\quad$ or $\quad \dfrac{dp}{p(p-3)} = dx$ $\quad\Rightarrow\quad \displaystyle\int \frac{dp}{p(p-3)} = \int dx + k$

But $\quad \dfrac{1}{p(p-3)} = \dfrac{-1}{3}\cdot\dfrac{1}{p} + \dfrac{1}{3}\cdot\dfrac{1}{p-3}$, by partial fractions.

$\therefore \quad \displaystyle\int \frac{dp}{p(p-3)} = \frac{1}{3}\log\left(\frac{p-3}{p}\right)$ and hence we have

$$\frac{1}{3}\log\left(\frac{p-3}{p}\right) = x + k \quad \text{or} \quad \log\left(\frac{p-3}{p}\right) = 3x + 3k$$

Hence, $\dfrac{p-3}{p} = e^{3x+3k}$ $\quad$ or $\quad 1 - \dfrac{3}{p} = c\,e^{3x}$ $\quad$ where $\quad c = e^{3k}$

ie., $\quad 1 - c\,e^{3x} = \dfrac{3}{p}$ $\quad$ or $\quad p = \dfrac{3}{1 - c\,e^{3x}}$

Thus by using this value of p in (1) we obtain the general solution

$$y = 3x + \log\left[\frac{3}{1 - ce^{3x}}\right]$$

Note : *We can also notice that the given equation is solvable for x and alternative form of solution is as follows.*

$y = 3x + \log p$ $\quad\ldots(1)$

Differentiating w.r.t y

$$1 = 3\cdot\frac{1}{p} + \frac{1}{p}\frac{dp}{dy} \quad \text{or} \quad \left(1 - \frac{3}{p}\right)p = \frac{dp}{dy}$$

ie., $\quad p - 3 = \dfrac{dp}{dy}$ $\quad$ or $\quad \dfrac{dp}{p-3} = dy$ $\quad\Rightarrow\quad \displaystyle\int \frac{dp}{p-3} = \int dy + k$

ie., $\quad \log(p-3) = y + k$ $\quad$ or $\quad p - 3 = e^{y+k}$ $\quad$ or $\quad p = 3 + ce^{y}$, $\quad$ where $e^{k} = c$

Using this value of p in (1) we obtain the general solution.

$$\mathbf{y = 3x + \log(3 + c\,e^{y})}$$

---

17. ***Solve :*** $p^2 + 4x^5 p - 12x^4 y = 0$. ***Obtain the singular solution also.***

>> The given equation is solvable for y only. Imp

$$p^2 + 4x^5 p - 12x^4 y = 0 \qquad \ldots (1)$$

ie., $$y = \frac{p^2 + 4x^5 p}{12x^4} = f(x, p)$$

Differentiating (1) w.r.t x,

$$2p\frac{dp}{dx} + 4x^5\frac{dp}{dx} + 20x^4 p - 12x^4 p - 48x^3 y = 0$$

ie., $$(2p + 4x^5)\frac{dp}{dx} + (8x^4 p - 48x^3 y) = 0$$

ie., $$(2p + 4x^5)\frac{dp}{dx} + 8x^3(xp - 6y) = 0$$

ie., $$(2p + 4x^5)\frac{dp}{dx} + 8x^3\left[xp - \frac{p^2 + 4x^5 p}{2x^4}\right] = 0$$

ie., $$(2p + 4x^5)\frac{dp}{dx} + \frac{4}{x}[2x^5 p - p^2 - 4x^5 p] = 0$$

ie., $$(2p + 4x^5)\frac{dp}{dx} + \frac{4}{x}(-p^2 - 2x^5 p) = 0$$

or $$(p + 2x^5)\frac{dp}{dx} = \frac{2p}{x}(p + 2x^5)$$

ie., $$\frac{dp}{dx} - \frac{2p}{x} = 0 \quad \text{or} \quad \frac{dp}{2p} = \frac{dx}{x} \quad \Rightarrow \quad \frac{1}{2}\int\frac{dp}{p} = \int\frac{dx}{x} + k$$

ie., $$\log\sqrt{p} - \log x = k \quad \text{or} \quad \log(\sqrt{p}/x) = \log c \quad \Rightarrow \quad \sqrt{p}/x = c$$

We have $\sqrt{p}/x = c$ or $p = c^2 x^2$

Using $p = c^2 x^2$ in (1) we have,

$$c^4 x^4 + 4x^5 \cdot c^2 x^2 - 12x^4 y = 0 \quad \text{or} \quad c^4 + 4c^2 x^3 = 12y$$

Thus by setting $c^2 = k$, **the general solution is** $\mathbf{k^2 + 4kx^3 = 12y}$

Further, differentiatig partially w.r.t. k we have,

$$2k + 4x^3 = 0 \quad \text{or} \quad k = -2x^3$$

Using $k = -2x^3$ in the general solution we have,

$$4x^6 - 8x^6 = 12y \quad \text{or} \quad -4x^6 = 12y \quad \text{or} \quad 3y + x^6 = 0$$

Thus $\boldsymbol{x^6 + 3y = 0}$ **is the singular solution.**

---

18. *Obtain the general solution and the singular solution of the equation* $x^4 p^2 + 2x^3 py - 4 = 0$

>> The given equation is solvable for y only.

$$x^4 p^2 + 2x^3 py - 4 = 0 \qquad \ldots (1)$$

Differentiating w.r.t. x,

$$2x^4 p \frac{dp}{dx} + 4x^3 p^2 + 2x^3 p^2 + 6x^2 py + 2x^3 y \frac{dp}{dx} = 0$$

ie., $$2x^3 \frac{dp}{dx} (xp + y) + (6x^3 p^2 + 6x^2 py) = 0$$

ie., $$2x^3 \frac{dp}{dx} (xp + y) + 6x^2 p (xp + y) = 0$$

or $$2x^3 \frac{dp}{dx} + 6x^2 p = 0 \quad \text{or} \quad x \frac{dp}{dx} + 3p = 0$$

ie., $$x \frac{dp}{dx} = -3p \quad \text{or} \quad \frac{dp}{p} = -3 \frac{dx}{x} \quad \Rightarrow \quad \int \frac{dp}{p} + 3 \int \frac{dx}{x} = k$$

ie., $$\log p + 3 \log x = k \quad \text{or} \quad \log (px^3) = \log c \quad \Rightarrow \quad px^3 = c$$

Using $p = c/x^3$ in (1) we have,

$$x^4 \cdot \frac{c^2}{x^6} + 2x^3 y \cdot \frac{c}{x^3} - 4 = 0 \quad \text{or} \quad \frac{c^2}{x^2} + 2cy - 4 = 0$$

Thus $\boldsymbol{c^2 + 2c x^2 y = 4x^2}$ is the general solution.

Now differentiating partially w.r.t c we have, $2c + 2x^2 y = 0$ or $c = -x^2 y$

Hence the general solution becomes

$$x^4 y^2 - 2x^4 y^2 = 4x^2 \quad \text{or} \quad -x^4 y^2 = 4x^2$$

Thus $\boldsymbol{x^2 y^2 + 4 = 0}$ is the singular solution.

---

19. *Obtain the general solution of the equation* $xp^4 - 2yp^3 + 12x^3 = 0$ *and hence show that* $8\,x^3 = 3\,y^2$ *is the singular solution.*

>> The given equation is solvable for y only.

$$xp^4 - 2yp^3 + 12x^3 = 0 \qquad \ldots(1)$$

ie., $$y = \frac{xp^4 + 12x^3}{2p^3} = f(x\,,\,p)$$

Differentiating (1) w.r.t. x,

$$4xp^3\frac{dp}{dx} + p^4 - 6yp^2\frac{dp}{dx} - 2\,p^4 + 36x^2 = 0$$

ie., $$2p\frac{dp}{dx}\,(\,2xp^2 - 3yp\,) = p^4 - 36x^2$$

ie., $$2p\frac{dp}{dx}\left[\,2xp^2 - 3p\cdot\frac{xp^4 + 12x^3}{2p^3}\,\right] = p^4 - 36x^2$$

ie., $$2p\frac{dp}{dx}\left[\,2xp^2 - \frac{3\,(\,xp^4 + 12x^3\,)}{2p^2}\,\right] = p^4 - 36x^2$$

ie., $$2p\frac{dp}{dx}\left[\,\frac{xp^4 - 36x^3}{2p^2}\,\right] = p^4 - 36x^2$$

ie., $$\frac{x\,(\,p^4 - 36x^2\,)}{p}\,\frac{dp}{dx} = (\,p^4 - 36\,x^2\,)$$

or $$\frac{x}{p}\,\frac{dp}{dx} = 1 \quad\text{or}\quad \frac{dp}{p} = \frac{dx}{x} \quad\Rightarrow\quad \int\frac{dp}{p} - \int\frac{dx}{x} = k$$

ie., $$\log p - \log x = \log c \quad\text{or}\quad \log\,(\,p/x\,) = \log c \quad\Rightarrow\quad p = c\,x$$

Using $p = cx$ in (1) we have,

$$c^4\,x^5 - 2yc^3\,x^3 + 12x^3 = 0 \quad\text{or}\quad c^4\,x^2 - 2yc^3 + 12 = 0$$

Thus the **general solution is** $\mathbf{c^4\,x^2 + 12 = 2yc^3}$

Now, differentiating partially w.r.t c we have,

$$4c^3\,x^2 = 6c^2\,y \quad\text{or}\quad c = 3y/2x^2$$

Using $c = 3y/2x^2$ in the general solution we have

$$\frac{81\,y^4}{16\,x^8}\cdot x^2 + 12 = 2y\,\frac{27y^3}{8x^6} \quad\text{or}\quad \frac{81\,y^4}{16\,x^6} + 12 = \frac{27\,y^4}{4x^6}$$

ie., $81y^4 + 192x^6 = 108y^4$ or $192x^6 = 27y^4$ or $64x^6 = 9y^4$

ie., $64\,x^6 = 9y^4 \Rightarrow \sqrt{64\,x^6} = \sqrt{9y^4}$ or $8x^3 = 3y^2$

Thus the singular solution is $8x^3 = 3y^2$.

---

20. *Solve :* $y = 2px + p^2 y$ Imp

>> The given equation is solvable for p or y or x. But things get complicated in the first two cases and we proceed to obtain the solution of the equation as solvable for x.

$$y = 2px + p^2 y \qquad \ldots (1)$$

That is, $y - p^2 y = 2px$ or $x = \dfrac{y(1-p^2)}{2p} = f(y, p)$

Differentiating (1) w.r.t. y

$$1 = 2p \cdot \frac{1}{p} + 2x\frac{dp}{dy} + p^2 + 2p\frac{dp}{dy}y$$

ie., $$-(1+p^2) = 2\frac{dp}{dy}(x+py)$$

ie., $$-(1+p^2) = 2\frac{dp}{dy}\left[\frac{y(1-p^2)}{2p} + py\right]$$

ie., $$-(1+p^2) = 2y\frac{dp}{dy}\left[\frac{1-p^2+2p^2}{2p}\right]$$

ie., $$-(1+p^2) = \frac{y}{p}\frac{dp}{dy}(1+p^2)$$

ie., $$-1 = \frac{y}{p}\frac{dp}{dy} \quad \text{or} \quad \frac{dp}{p} + \frac{dy}{y} = 0 \quad \Rightarrow \quad \int\frac{dp}{p} + \int\frac{dy}{y} = k$$

ie., $$\log(py) = k \quad \text{or} \quad \log(py) = \log c \quad \Rightarrow \quad py = c$$

Using $p = c/y$ in (1) we have,

$$y = \frac{2c}{y}\cdot x + \frac{c^2}{y^2}\cdot y \quad \text{or} \quad y = \frac{2cx}{y} + \frac{c^2}{y}$$

Thus $y^2 = 2cx + c^2$ is the general solution.

Note : Singular Solution

Differentiating $y^2 = 2cx + c^2$ partially w.r.t c we obtain $2x + 2c = 0$ or $c = -x$

Hence the general solution becomes, $y^2 = -2x^2 + x^2$

Thus **$x^2 + y^2 = 0$ is the singular solution.**

---

21. ***Obtain the general and singular solution of the equation*** $y^2 \log y = xpy + p^2$

>> The given equation is solvable for x only.

$$y^2 \log y = xpy + p^2 \qquad \ldots (1)$$

ie., $$x = \frac{y^2 \log y - p^2}{py} = f(y, p)$$

Differentiating (1) w.r.t y,

$$y^2 \cdot \frac{1}{y} + 2y \log y = xp + py \cdot \frac{1}{p} + xy \frac{dp}{dy} + 2p \frac{dp}{dy}$$

ie., $$y + 2y \log y = xp + y + xy \frac{dp}{dy} + 2p \frac{dp}{dy}$$

ie., $$2y \log y - xp = \frac{dp}{dy} (xy + 2p)$$

ie., $$2y \log y - \frac{y^2 \log y - p^2}{y} = \frac{dp}{dy} \left[\frac{y^2 \log y - p^2}{p} + 2p \right]$$

ie., $$\frac{(y^2 \log y + p^2)}{y} = \frac{dp}{dy} \frac{(y^2 \log y + p^2)}{p}$$

ie., $$\frac{1}{y} = \frac{1}{p} \frac{dp}{dy} \quad \text{or} \quad \frac{dy}{y} = \frac{dp}{p} \quad \Rightarrow \quad \int \frac{dy}{y} = \int \frac{dp}{p} + k$$

ie., $$\log y = \log p + \log c \quad \text{or} \quad \log y = \log (cp) \quad \Rightarrow \quad y = cp \quad \text{or} \quad p = y/c$$

Using $p = y/c$ in (1) we have,

$$y^2 \log y = x \cdot \frac{y}{c} \cdot y + \frac{y^2}{c^2} \quad \text{or} \quad \log y = \frac{x}{c} + \frac{1}{c^2}$$

Thus the **general solution is $c^2 \log y = c\,x + 1$**

Now differentiating partially w.r.t c we have,

$$2\,c \log y = x \quad \text{or} \quad c = \frac{x}{2 \log y}$$

Hence the general solution becomes,

$$\frac{x^2}{4 (\log y)^2} \cdot \log y = \frac{x}{2 \log y} \cdot x + 1$$

ie., $$\frac{x^2}{4 \log y} = \frac{x^2}{2 \log y} + 1 \quad \text{or} \quad x^2 = 2x^2 + 4 \log y$$

Thus $x^2 + 4 \log y = 0$ **is the singular solution.**

22. *Solve* : $p^3 - 4xyp + 8y^2 = 0$ Imp

>> The given equation is solvable for x only.

$$p^3 - 4xyp + 8y^2 = 0 \qquad \ldots (1)$$

$$x = \frac{p^3 + 8y^2}{4yp} = f(y, p)$$

Differentiating (1) w.r.t. y,

$$3p^2 \frac{dp}{dy} - 4xy\frac{dp}{dy} - 4yp \cdot \frac{1}{p} - 4px + 16y = 0$$

ie., $$\frac{dp}{dy}(3p^2 - 4xy) = 4px - 12y$$

ie., $$\frac{dp}{dy}\left[3p^2 - \frac{p^3 + 8y^2}{p}\right] = \left[\frac{p^3 + 8y^2}{y} - 12y\right]$$

ie., $$\frac{dp}{dy}\left[\frac{2p^3 - 8y^2}{p}\right] = \frac{p^3 - 4y^2}{y}$$

ie., $$\frac{2}{p}\frac{dp}{dy}(p^3 - 4y^2) = \frac{(p^3 - 4y^2)}{y}$$

ie., $$\frac{2}{p}\frac{dp}{dy} = \frac{1}{y} \quad \text{or} \quad 2\frac{dp}{p} = \frac{dy}{y} \quad \Rightarrow \quad 2\int\frac{dp}{p} = \int\frac{dy}{y} + k$$

ie., $$2\log p = \log y + \log c \quad \text{or} \quad \log(p^2) = \log(cy) \quad > \quad p^2 = cy \quad \text{or} \quad p = \sqrt{cy}$$

Using $p = \sqrt{cy}$ in (1) we have,

$$cy\sqrt{cy} - 4xy\sqrt{cy} + 8y^2 = 0$$

Dividing throughout by $y\sqrt{y} = y^{3/2}$ we have,

$$c\sqrt{c} - 4x\sqrt{c} + 8\sqrt{y} = 0$$

ie., $$\sqrt{c}(c - 4x) = -8\sqrt{y}$$

Thus the general solution is $c(c - 4x)^2 = 64y$

---

23. *Solve* : $y = 2px + y^2p^3$

>> The given equation is solvabale for x only.

$$y = 2px + y^2p^3 \qquad \ldots (1)$$

ie., $$x = \frac{y(1 - yp^3)}{2p} = f(y, p)$$

Differentiating (1) w.r.t y,

$$1 = 2p \cdot \frac{1}{p} + 2\frac{dp}{dy}x + 3y^2p^2\frac{dp}{dy} + 2yp^3$$

ie., $$-(1+2yp^3) = \frac{dp}{dy}(2x+3y^2p^2)$$

ie., $$-(1+2yp^3) = \frac{dp}{dy}\left[\frac{y(1-yp^3)}{p} + 3y^2p^2\right]$$

ie., $$-(1+2yp^3) = \frac{dp}{dy}\left[\frac{y+2y^2p^3}{p}\right]$$

ie., $$-(1+2yp^3) = \frac{dp}{dy}\frac{y(1+2yp^3)}{p}$$

ie., $$-1 = \frac{y}{p}\frac{dp}{dy} \quad \text{or} \quad -\frac{dy}{y} = \frac{dp}{p} \quad \Rightarrow \quad \int\frac{dy}{y} + \int\frac{dp}{p} = k$$

ie., $\log y + \log p = \log c$ or $\log(yp) = \log c$ $\Rightarrow$ $yp = c$ or $p = c/y$

Using $p = c/y$ in (1) we have

$$y = 2\frac{c}{y}x + y^2\frac{c^3}{y^3} \quad \text{or} \quad y = 2\frac{cx}{y} + \frac{c^3}{y}$$

Thus the general solution is $y^2 = 2cx + c^3$

24. *Solve* : $p = \tan\left(x - \frac{p}{1+p^2}\right)$

>> The give equation in the equivalent form is,

$$\tan^{-1}p = x - \frac{p}{1+p^2}$$

or $$x = \tan^{-1}p + \frac{p}{1+p^2} \qquad \ldots(1)$$

Differentiating (1) w.r.t. y,

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{1-p^2}{(1+p^2)^2}\frac{dp}{dy}$$

ie., $$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy}\left[1 + \frac{1-p^2}{1+p^2}\right]$$

ie., $\frac{1}{p}\cdot\frac{dp}{dy} = \frac{2}{(1+p^2)^2}$

ie., $dy = \frac{2p\,dp}{(1+p^2)^2} \Rightarrow \int dy = \int \frac{2\,p\,dp}{(1+p^2)^2} + c$

ie., $y = \int \frac{2\,p\,dp}{(1+p^2)^2} + c$

Using $(1+p^2) = u$, we have $2p\,dp = du$

ie., $y = \int \frac{du}{u^2} + c$ or $y = \frac{-1}{u} + c$ or $y = \frac{-1}{1+p^2} + c$

Thus $y + \frac{1}{1+p^2} = c$ & $x = \tan^{-1} p + \frac{p}{1+p^2}$ **represents the general solution.**

1.26 Clairaut's Equation

The equation of the form

$$y = px + f(p) \qquad \ldots (1)$$

is known as ***Clairaut's equation.***

This being in the form $y = F(x, p)$, that is solvable for y, we differentiate (1) w.r.t. x

$\therefore \quad \frac{dy}{dx} = p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$

or $\quad 0 = \frac{dp}{dx}[x + f'(p)]$

This implies that $\frac{dp}{dx} = 0$ and hence $p = c$

Using $p = c$ in (1) we obtain the **general solution of clairaut's equation** in the form

$$y = cx + f(c) \qquad \ldots (2)$$

Note :

1. *The general solution of clairaut's equation is obtained by simply replacing p by c.*
2. *The solution can be written instantly when the equation is in clairaut's form or when the equation is put into the clairaut's form.*
3. *We can also obtain the singular solution.*

WORKED PROBLEMS

25. *Solve :* $y = px + \dfrac{a}{p}$

>> The given equation is Clairaut's equation of the form $y = px + f(p)$, whose general solution is $y = cx + f(c)$

Thus **the general solution is** $y = cx + \dfrac{a}{c}$

Note : ***Singular solution***

Differentiating partially w.r.t c *we have,*

$$0 = x - \frac{a}{c^2} \quad \text{or} \quad \frac{a}{c^2} = x \quad \text{or} \quad c^2 = \frac{a}{x} \quad > \quad c = \sqrt{\frac{a}{x}}$$

Hence $y = cx + (a/c)$ *becomes,*

$$y = \sqrt{a/x} \cdot x + a\sqrt{x/a} \quad \text{or} \quad y = 2\sqrt{ax}$$

Thus $y^2 = 4ax$ **is the singular solution.**

---

26. *Obtain the general and singular solution of the equation* $p = \log(px - y)$

>> $p = \log(px - y)$, by data .

or $\quad e^p = px - y \quad$ or $\quad y = px - e^p$

Since $y = px - e^p$ is in the form of Clairaut's equation $y = px + f(p)$, the general solution is $y = cx + f(c)$.

Thus the **general solution is** $y = cx - e^c$

Now differentiating partially w.r.t c we have,

$$0 = x - e^c \quad \text{or} \quad e^c = x \quad > \quad c = \log x$$

Thus the **singular solution is** $y = x\log x - x$.

---

27. *Solve :* $(p+1)^2(y - px) = 1$

>> We have, $y - px = \dfrac{1}{(p+1)^2}$ or $y = px + \dfrac{1}{(p+1)^2}$

This equation being in the form of Clairaut's equation $y = px + f(p)$, the general solution is $y = cx + f(c)$

Thus the general solution is $y = cx + \dfrac{1}{(c+1)^2}$

---

28. Show that the equation $xp^2 + px - py + 1 - y = 0$ is Clairaut's equation. Hence obtain the general and singular solution.

>> $xp^2 + px - py + 1 - y = 0$, by data

ie., $$xp^2 + px + 1 = y(p+1)$$

ie., $$y = \frac{xp(p+1)+1}{p+1}$$

or $$y = px + \frac{1}{p+1} \qquad \ldots (1)$$

(1) is in the Clairaut's form $y = px + f(p)$ whose general solution is $y = cx + f(c)$

Thus the **general solution is** $y = cx + \dfrac{1}{c+1}$

Now differentiating partially w.r.t c we have,

$$0 = x - \frac{1}{(c+1)^2} \quad \text{or} \quad (c+1)^2 = \frac{1}{x} \quad > \quad (c+1) = \frac{1}{\sqrt{x}} \quad \text{or} \quad c = \frac{1}{\sqrt{x}} - 1$$

Hence the general solution becomes

$$y = \left(\frac{1}{\sqrt{x}} - 1\right)x + \sqrt{x}$$

ie., $$y = \sqrt{x} - x + \sqrt{x} \quad \text{or} \quad x + y = 2\sqrt{x}$$

Thus the singular solution is $(x+y)^2 = 4x$

Remark : *We can also obtain solution in the method : Solvable for y.*

---

29. Modify the following equation into Clairaut's form. Hence obtain the associated general and singular solutions

$$xp^2 - py + kp + a = 0$$

>> $x p^2 - py + k p + a = 0$, by data

ie., $$x p^2 + k p + a = py$$

ie., $$y = \frac{p(xp+k)+a}{p} \quad \text{or} \quad y = xp + k + \frac{a}{p}$$

ie., $$y = px + \left(k + \frac{a}{p}\right) \qquad \ldots (1)$$

Here (1) is in the Clairaut's form $y = px + f(p)$ whose general solution is $y = cx + f(c)$

Thus the general solution is $y = cx + \left(k + \frac{a}{c}\right)$

Now differentiating partially w.r.t c we have,

$$0 = x - \frac{a}{c^2} \quad \text{or} \quad \frac{a}{c^2} = x \quad \text{or} \quad c^2 = \frac{a}{x} \quad \Rightarrow \quad c = \sqrt{a/x}$$

Hence the general solution becomes,

$$y = \sqrt{a/x}\; x + k + a\sqrt{x/a} \quad \text{or} \quad y - k = 2\sqrt{ax}$$

Thus the singular solution is $(y - k)^2 = 4ax$.

Remark : *We can also obtain the solution in the method : Solvable for* y.

30. *Obtain the general solution and the singular solution of the following equation as Clairaut's equation :* $xp^3 - yp^2 + 1 = 0$

>> $xp^3 - yp^2 + 1 = 0$ by data.

ie., $yp^2 = xp^3 + 1$ or $y = \frac{xp^3 + 1}{p^2}$

ie., $y = px + \frac{1}{p^2}$ is in the Clairaut's form $y = px + f(p)$ whose general solution is $y = cx + f(c)$

Thus the general solution is $y = cx + \frac{1}{c^2}$

Now, differentiating partially w.r.t, c we have,

$$0 = x - \frac{2}{c^3} \quad \text{or} \quad c^3 = \frac{2}{x} \quad \Rightarrow \quad c = \left(\frac{2}{x}\right)^{1/3}$$

Hence the general solution becomes,

$$y = \left(\frac{2}{x}\right)^{1/3} x + \left(\frac{x}{2}\right)^{2/3} \quad \text{or} \quad y = 2^{1/3} \,.\, x^{2/3} + \frac{x^{2/3}}{2^{2/3}}$$

ie., $2^{2/3}\, y = 2x^{2/3} + x^{2/3}$ or $2^{2/3}\, y = 3x^{2/3}$

Equivalently, $(2^{2/3}\,y)^3 = (3x^{2/3})^3$ or $4y^3 = 27x^2$

Thus the singular solution is $4y^3 = 27x^2$

---

31. *Solve the equation $(px - y)(py + x) = 2p$ by reducing into Clairaut's form, taking the substitutions $X = x^2$, $Y = y^2$*

Imp

>> $X = x^2 \;\;\to\;\; \dfrac{dX}{dx} = 2x$

$$Y = y^2 \;\;\to\;\; \frac{dY}{dy} = 2y$$

Now, $p = \dfrac{dy}{dx} = \dfrac{dy}{dY}\,\dfrac{dY}{dX}\,\dfrac{dX}{dx}$ and let $P = \dfrac{dY}{dX}$

ie., $p = \dfrac{1}{2y}\cdot P\cdot 2x$ or $p = \dfrac{x}{y}\,P$

ie., $p = \dfrac{\sqrt{X}}{\sqrt{Y}}\,P$

Consider $(px - y)(py + x) = 2p$

ie., $$\left[\frac{\sqrt{X}}{\sqrt{Y}}\,P\sqrt{X} - \sqrt{Y}\right]\left[\frac{\sqrt{X}}{\sqrt{Y}}\,P\sqrt{Y} + \sqrt{X}\right] = 2\,\frac{\sqrt{X}}{\sqrt{Y}}\,P$$

or $$\frac{(PX - Y)}{\sqrt{Y}}\,(P + 1)\sqrt{X} = 2\,\frac{\sqrt{X}}{\sqrt{Y}}\,P$$

ie., $(PX - Y)(P + 1) = 2P$ or $PX - Y = \dfrac{2P}{P+1}$

ie., $Y = PX - \dfrac{2P}{P+1}$ is in the Clairaut's form and hence the associated general solution is

$$Y = cX - \frac{2c}{c+1} \quad\text{or}\quad y^2 = c\,x^2 - \frac{2c}{c+1}$$

Thus the required general solution of the given equation is $y^2 = c\,x^2 - \dfrac{2c}{c+1}$

---

32. *Solve :* $e^{4x}(p-1)+e^{2y}p^2=0$ *by using the substitution* [illegible] *and* [illegible]

$$u = e^{2x} \quad \Rightarrow \quad \frac{du}{dx} = 2e^{2x}$$

$$v = e^{2y} \quad \Rightarrow \quad \frac{dv}{dy} = 2e^{2y}$$

Now, $p = \dfrac{dy}{dx} = \dfrac{dy}{dv}\,\dfrac{dv}{du}\,\dfrac{du}{dx}$ and let $P = \dfrac{dv}{du}$

ie., $p = \dfrac{1}{2e^{2y}}\,P\,2e^{2x}$ or $p = \dfrac{u}{v}P$

Consider $e^{4x}(p-1)+e^{2y}p^2 = 0$

ie., $u^2\left(\dfrac{u}{v}P-1\right)+v\cdot\dfrac{u^2}{v^2}P^2 = 0$

ie., $u^2\dfrac{(uP-v)}{v}+\dfrac{u^2P^2}{v} = 0$

or $uP-v+P^2 = 0$

ie., $v = Pu+P^2$ is in the Clairaut's form and hence the associated general solution is

$v = cu+c^2$ or $e^{2y} = ce^{2x}+c^2$

Thus the required general solution of the given equation is $e^{2y} = ce^{2x}+c^2$

33. *Find the general* [illegible] *solution of the equation* [illegible] *reducing into Clairaut's form* [illegible]

>> This problem is similar to Problem-31. Proceeding on the same lines we have $p = \dfrac{\sqrt{X}}{\sqrt{Y}}P$, where $P = \dfrac{dY}{dX}$ the given equation $x^2(y-px) = p^2y$ becomes,

$$X\left(\sqrt{Y}-\frac{\sqrt{X}}{\sqrt{Y}}P\sqrt{X}\right) = \frac{X}{Y}P^2\sqrt{Y}$$

ie., $X\dfrac{(Y-PX)}{\sqrt{Y}} = \dfrac{XP^2}{\sqrt{Y}}$ or $Y-PX = P^2$

ie., $Y = PX+P^2$ is in the Clairaut's form.

The associatead general solution is $Y = cX+c^2$

Thus the required general solution of the given equation is $y^2 = cx^2+c^2$

Now differentiating w.r.t c partially we have,

$$0 = x^2 + 2c \quad \text{or} \quad c = -x^2/2$$

Hence the general solution becomes,

$$y^2 = \frac{-x^4}{2} + \frac{x^4}{4} \quad \text{or} \quad 4y^2 = -x^4$$

Thus the required singular solution is $x^4 + 4y^2 = 0$

34. *Solve :* $(y - px)^2 = 4p^2 + 9$

>> We have, $(y - px)^2 = 4p^2 + 9$

$$\Rightarrow \quad y - px = \sqrt{4p^2 + 9}$$

or $\quad y = px + \sqrt{4p^2 + 9}$

This equation being in the Clairaut's form, $y = px + f(p)$ has the general solution $y = cx + f(c)$

Thus the required general solution is $y = cx + \sqrt{4c^2 + 9}$

35. *Solve :* $p = \cos y \cos px + \sin y \sin px$

>> $p = \cos y \cos px + \sin y \sin px$, by data.

ie., $\quad p = \cos(y - px) \quad$ or $\quad \cos^{-1} p = y - px$

ie., $\quad y = px + \cos^{-1} p$ is in the Clairaut's form $y = px + f(p)$ whose general solution is $y = cx + f(c)$

Thus the required general solution is $y = cx + \cos^{-1} c$

EXERCISES

Solve the following equations

1. $\left(\frac{dy}{dx}\right)^2 - 5\left(\frac{dy}{dx}\right) + 6 = 0$

2. $x^2\left(\frac{dy}{dx}\right)^2 + xy\left(\frac{dy}{dx}\right) - 6y^2 = 0$

3. $x^2 p^2 + 3xyp + 2y^2 = 0$

4. $xy p^2 + (3x^2 - 2y^2)p - 6xyp = 0$

5. $y + px = p^2 x^4$; Find the singular solution also.

6. $y + p^2 = 2px$.

7. $xp^2 - 2yp + ax = 0$; Find the singular solution also.

8. $y = 3px + 6p^2 y^2$; Find the singular solution also.

9. $(y - px) (p - 1) = p$

10. $y = px + (2/p)$

11. $p + \cos y \sin px = \sin y \cos px$

12. Solve : $e^{3x} (p - 1) + p^3 e^{2y} = 0$ by using the substitution $u = e^x, v = e^y$

ANSWERS

1. $(y - 2x - c) (y - 3x - c) = 0$

2. $(y - cx^2) (x^3 y - c) = 0$

3. $(xy - c) (x^2 y - c) = 0$

4. $(y - cx^2) (y^2 + 3x^2 - c) = 0$

5. $xy + c = c^2 x \;\; ; \;\; 4x^2 y + 1 = 0$

6. $x = \frac{2}{3} p + \frac{c}{p^2}$ and $y = \frac{p^2}{3} + \frac{2c}{p}$

7. $2cy = c^2 x^2 + a \;\; ; \;\; y = ax^2$

8. $y^3 = 3cx + 6c^2 \;\; ; \;\; 3x^2 + 8y^3 = 0$

9. $y = cx + \frac{c}{c - 1}$

10. $y = cx + \frac{2}{c}$

11. $y = cx + \sin^{-1} c$

12. $e^y = c\, e^x + c^2$

1.3 Applications of Differential Equations of First Order

We are familiar with an application of differential equation of first order in the form of *Orthogonal Trajectories.*

We present a few illustrative problems on *Application to Electric Circuits.*

A governing first order differential equation by Kirchhoff's law is given by

$$L\frac{di}{dt}+Ri = E$$

where L is the inductance, R is the resistance and E is the electromotive force.

ILLUSTRATIVE PROBLEMS

1. *A constant electromotive force E volts is applied to a circuit containing a constant resistance R ohms in series and a constant inductance L henries. If the initial current is zero, find the current in the circuit at any time t.*

>> We have the governing differential equation,

$$L\frac{di}{dt}+Ri = E \quad \text{or} \quad \frac{di}{dt}+\frac{R}{L}i = \frac{E}{L} \qquad \dots(1)$$

This is a linear d.e of the form $\frac{dy}{dx}+Py = Q$ whose solution is

$$y\,e^{\int P\,dx} = \int Q\,e^{\int P\,dx}\,dx+c$$

Hence the solution of (1) is given by,

$$i\,e^{Rt/L} = \int \frac{E}{L}\,e^{Rt/L}\,dt+c$$

ie., $$i\,e^{Rt/L} = \frac{E}{L}\,\frac{e^{Rt/L}}{(R/L)}+c$$

ie., $$i\,e^{Rt/L} = \frac{E}{R}\,e^{Rt/L}+c$$

or $$i = \frac{E}{R}+c\,e^{-Rt/L} \qquad \dots(2)$$

Using the initial condition, $i = 0$ when $t = 0$, (2) becomes,

$$0 = \frac{E}{R}+c \quad \text{or} \quad c = -\frac{E}{R}$$

Using this value of c in (2), we obtain the current in the circuit at any time t,

$$\boldsymbol{i = \frac{E}{R}\left[1-e^{-Rt/L}\right]}$$

---

2. *A voltage $E e^{-at}$ is applied at $t = 0$ to a circuit containing inductance L and resistance R. Determine the current at any time t.*

>> We have the governing differential equation,

$$L\frac{di}{dt} + Ri = E \qquad \ldots(1)$$

Using the data, $L\frac{di}{dt} + Ri = Ee^{-at}$

ie., $$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}e^{-at}$$

The solution of this linear equation is

$$i\,e^{Rt/L} = \int \frac{E}{L}e^{-at}e^{Rt/L}\,dt + c$$

ie., $$i\,e^{Rt/L} = \frac{E}{L}\int e^{(R/L-a)t}\,dt + c$$

ie., $$i\,e^{Rt/L} = \frac{E}{L}\,\frac{e^{(R/L-a)t}}{(R/L-a)} + c$$

ie., $$i\,e^{Rt/L} = \frac{E}{R-aL}\,e^{(R/L-a)t} + c$$

or $$i = \frac{Ee^{-at}}{R-aL} + c\,e^{-Rt/L} \qquad \ldots(2)$$

Using the initial condition, $i = 0$ when $t = 0$,

$$0 = \frac{E}{R-aL} + c \quad \text{or} \quad c = \frac{-E}{R-aL}$$

Using this value of c in (2), we obtain the current at any time t.

$$\mathbf{i = \frac{E}{R-aL}\,(e^{-at} - e^{-Rt/L})}.$$

---

3. *The current i amperes at any time t is given by $L\frac{di}{dt} + Ri = E$, when a resistance R ohms is connected in series with an inductance L henries and e.m.f of E volts. If $E = 10\sin t$ volts and $i = 0$ when $t = 0$, find i as a function of t.*

>> We have $L\frac{di}{dt} + Ri = 10\sin t$, using the data.

ie., $$\frac{di}{dt} + \frac{R}{L}i = \frac{10}{L}\sin t \qquad \ldots(1)$$

The solution of this linear equation is,

$$i\,e^{Rt/L} = \int \frac{10}{L}\sin t\, e^{Rt/L}\,dt + c$$

ie., $$i\,e^{Rt/L} = \frac{10}{L}\int e^{Rt/L}\sin t\;dt + c$$

Using $\int e^{at}\sin bt\,dt = \frac{e^{at}}{a^2+b^2}\,(a\sin bt - b\cos bt)$ we have,

$$i\,e^{Rt/L} = \frac{10}{L}\,\frac{e^{Rt/L}}{(R/L)^2+1}\left(\frac{R}{L}\sin t - \cos t\right) + c$$

ie., $$i\,e^{Rt/L} = \frac{10}{R^2+L^2}\,e^{Rt/L}\,(R\sin t - L\cos t) + c$$

or $$i = \frac{10}{R^2+L^2}\,(R\sin t - L\cos t) + c\,e^{-Rt/L} \quad \ldots (2)$$

Using the initial condition, $i = 0$ when $t = 0$ we have,

$$0 = \frac{-10L}{R^2+L^2} + c \quad \text{or} \quad c = \frac{10L}{R^2+L^2}$$

Using this value of c in (2) we obtain i as a function of t.

$$\boldsymbol{i = \frac{10}{R^2+L^2}\left\{R\sin t - L\cos t + L\,e^{-Rt/L}\right\}}$$

- -

4. *The differential equation for the current i in an electric circuit containing an inductance L and a resistance R in series and acted on by an eletromotive force E sin wt satisfies the equation $L\frac{di}{dt} + Ri = E\sin wt$. Find the value of current at any time t, if initially there is no current in the circuit.*

\>> The given equation is put in the form,

$$\frac{di}{dt} + \frac{R}{L}\,i = \frac{E}{L}\sin wt \quad \ldots (1)$$

The solution of this linear equation is

$$i\,e^{Rt/L} = \int \frac{E}{L}\sin wt\, e^{Rt/L}\,dt + c$$

ie., $$i\,e^{Rt/L} = \frac{E}{L}\int e^{Rt/L}\sin wt\,dt + c$$

Note : *We can proceed as in the previous problem and obtain the solution. We can also obtain the solution using the following alternative formula for the integral of* e^{at} sin *bt.*

We have $\int e^{at} \sin bt \, dt = \dfrac{e^{at}}{\sqrt{a^2+b^2}} \sin(bt - \tan^{-1} b/a)$

$$\therefore \quad i\,e^{Rt/L} = \frac{E}{L} \frac{e^{Rt/L}}{\sqrt{R^2/L^2+w^2}} \sin[wt - \tan^{-1}(wL/R)] + c$$

ie.,

$$i\,e^{Rt/L} = \frac{Ee^{Rt/L}}{\sqrt{R^2+w^2L^2}} \sin[wt - \tan^{-1}(wL/R)] + c$$

Putting $\varphi = \tan^{-1}(wL/R)$ we have,

$$i = \frac{E}{\sqrt{R^2+w^2L^2}} \sin(wt - \varphi) + c\,e^{-Rt/L} \qquad \ldots (2)$$

Using the initial condition, $i = 0$ when $t = 0$, we have

$$0 = \frac{E \sin(-\varphi)}{\sqrt{R^2+w^2L^2}} + c \quad \text{or} \quad c = \frac{E \sin \varphi}{\sqrt{R^2+w^2L^2}}$$

Using this value of c in (2) we have,

$$i = \frac{E}{\sqrt{R^2+w^2L^2}} [\sin(wt-\varphi) + \sin\varphi\, e^{-Rt/L}], \quad \text{where} \quad \varphi = \tan^{-1}(wL/R)$$

- -

PART A

Unit - II

Differential Equations - 2

2.1 Introduction

We have discussed several methods for solving an ordinary differential equation of first order and first degree. In this chapter we discuss the method of solving differential equations (D.E) of second and higher orders but in first degree only.

The solution of a D.E of first order and first degree is arrived with direct involvement of integration, whereas integration is not directly involved in practical problems while arriving at the solution of a higher order equation. Here the complete solution is obtained by the application of several rules which are being established.

We notice that the solution of a first order and first degree equation is mostly in the implicit form $f(x, y) = c$. In the case of higher order equations the solution is always in the explicit form $y = f(x)$

In this unit we discuss the method of solving higher order differential equations involving derivatives with constant coefficients & solution of simultaneous differential equations of first order.

2.2 Linear differential equation of second and higher order with constant coefficients

Basically a D.E is said to be *linear* if the dependent variable and its derivatives are of first degree.

A D.E of the form,

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = \phi(x)$$

where $a_1, a_2, \cdots a_n$ are all constants is called a *linear differential equation of n^{th} order with constant coefficients.*

Using the familiar notation of differential operators :

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad \cdots D^n = \frac{d^n}{dx^n}$$

the differential equation can be put in the form,

$$[D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n] y = \phi(x)$$

ie., $f(D) y = \phi(x)$ $\quad\ldots(1)$

If $\phi(x) = 0$ the equation $f(D)y = 0$ is called a *homogeneous equation.*

If $\phi(x) \neq 0$ then the equation (1) is called a *non-homogeneous equation.*

$\boxed{2.21}$ Solution of homogeneous linear differential equation

In order to quickly grasp the conceptual content of the method of solving a homogeneous linear differential equation, the method is illustrated by taking a typical *second order equation* as the same can be conveniently extended for equations of order more than two.

Consider the equation in the form

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0 \quad a_1, a_2 \text{ are constants.} \qquad \ldots(1)$$

ie., $(D^2 + a_1 D + a_2) y = 0$ or $f(D) y = 0$ $\quad\ldots(2)$

where $f(D) = D^2 + a_1 D + a_2$

Firstly we are going to establish the following **fundamental property** :

If y_1 and y_2 are two linearly independent solutions (*one cannot be expressed in terms of the other*) of (2) then $c_1 y_1 + c_2 y_2$ is also a solution of (2) where c_1 and c_2 are arbitrary constants.

Since y_1 and y_2 are solutions of (2) we have,

$$f(D) y_1 = 0 \text{ and } f(D) y_2 = 0.$$

$$\therefore \quad f(D)[c_1 y_1 + c_2 y_2] = c_1 f(D) y_1 + c_2 f(D) y_2 = c_1 \cdot 0 + c_2 \cdot 0 = 0$$

Thus $f(D)[c_1 y_1 + c_2 y_2] = 0$ implies that $c_1 y_1 + c_2 y_2$ is a solution of (2)

Since the general solution of a second order differential equation has to contain two arbitrary constants $c_1 y_1 + c_2 y_2$ is the general solution of (2)

Denoting $y_c = c_1 y_1 + c_2 y_2$ we have, $f(D) y_c = 0$

y_c is called as the **Complimentary Function** (C.F) and it is the solution of the homogeneous D.E (1).

2.22 Solution of non-homogeneous linear differential equation

Let us consider $f(D)y = \phi(x)$ ...(1)

Also let $y = y_p(x)$ be a particular solution (*without the involvement of arbitrary constants*) of (1). Hence we must have,

$$f(D)y_p = \phi(x)$$

Now $f(D)[y_c+y_p] = f(D)y_c + f(D)y_p$

$$= 0+\phi(x) = \phi(x)$$

ie., $f(D)[y_c+y_p] = \phi(x)$

$\Rightarrow$ $y = y_c + y_p$ ...(2)

Thus y as given by (2) is also a solution of (1) and is called the **general solution** or **complete solution** of the given non-homogeneous equation.

The particular solution y_p is also called as the **Particular Integral (P.I)**

Remark : *The general solution of a non-homogeneous equation consists of two parts namely the complimentary function and the particular integral. The general solution is of the form* $y = C\cdot F+P\cdot I$ *or* $y = y_c+y_p$. *Obviously if* $\phi(x) = 0$, *the complimentary function itself is the general solution.*

2.23 Method of finding the complimentary function

Consider a second order homogeneous d.e

$$\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0 \quad \text{...(1)}$$

ie., $(D^2+a_1D+a_2)y = 0$ or $f(D)y = 0$ ...(2)

Taking $y = e^{mx}$ we have $Dy = m\,e^{mx}$, $D^2y = m^2e^{mx}$

Hence (2) becomes, $(m^2+a_1m+a_2)e^{mx} = 0$

$\Rightarrow$ $m^2+a_1m+a_2 = 0.$

This being a quadratic in m will have two roots which can be,

(i) real and distinct (ii) real and coincident (iii) complex

Case (i) Suppose the roots are m_1 and m_2, $m_1 \neq m_2$ then $y = e^{m_1 x}$ and $y = e^{m_2 x}$ are the two independent solutions of (1).

Hence by the fundamental property

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} \qquad \ldots (3)$$

Thus (3) is the general solution of (1).

Note : (1) *The equation* $m^2 + a_1 m + a_2 = 0$ *is* $f(m) = 0$ *and is called the* ***Auxilary Equation*** *(A.E)*

(2) *If* $m_1, m_2, m_3 \cdots m_n$ *are the n real and mutually distinct roots of the A.E associated with an* n^{th} *order equation then the general solution is given by*

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$$

Case (ii) Suppose the roots of the A.E are real and coincident.

ie., $m_1 = m_2 = m\,(say)$. Then $f(D)y = 0$ is of the form

$$(D-m)^2 y = 0 \quad ie., \; (D-m)(D-m)y = 0$$

Let $(D-m)y = t$, then this equation becomes $(D-m)t = 0$

ie., $\frac{dt}{dx} - mt = 0$ or $\frac{dt}{dx} = mt$

$\therefore$ $\frac{dt}{t} = m\,dx \Rightarrow \int \frac{dt}{t} = \int m\,dx + \alpha$, α is a constant.

ie., $\log t = mx + \alpha \Rightarrow t = e^{mx+\alpha} = e^{\alpha} e^{mx}$ or $t = k e^{mx}$

where $k = e^{\alpha}$ is a constant.

Further $t = (D-m)y = k e^{mx}$

ie., $\frac{dy}{dx} - my = ke^{mx}$ which is a linear equation in y of the form $\frac{dy}{dx} + Py = Q$

where $P = -m$, $Q = ke^{mx}$ and the solution is given by

$$y e^{\int P\,dx} = \int Q e^{\int P\,dx} dx + c$$

ie., $y e^{-mx} = \int k e^{mx} e^{-mx} dx + c$

ie., $y e^{-mx} = kx + c$ or $y = (kx + c) e^{mx}$

Thus we can say that the general solution of (1) in the case of real and coincident roots or repeated real roots is

$$y = (c_1 + c_2 x) e^{mx} \qquad \ldots (4)$$

Note : *If* $m_1 = m_2 = m_3 = \cdots m_n = m$ *be the roots of the A.E associated with an* n^{th} *order equation then* $y = (c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) e^{mx}$ *is the general solution.*

Case (iii) Suppose the roots of the A.E are complex which always occur in pairs say $p \pm iq$. Then the general solution of the d.e in the basic form is $y = a e^{(p+iq)x} + b e^{(p-iq)x}$ analogous to case (i) where a and b are arbitrary constants. This form can be simplified further.

ie., $y = e^{px} [a e^{iqx} + b e^{-iqx}]$

ie., $= e^{px} [a(\cos qx + i \sin qx) + b(\cos qx - i \sin qx)]$

$= e^{px} [(a+b)\cos qx + i(a-b)\sin qx]$

$\therefore$ $y = e^{px}(c_1 \cos qx + c_1{}' \sin qx)$ . . . (5)

where $c_1 = (a+b)$, $c_1{}' = i(a-b)$ are arbitrary constants.

However if the root pair is purely imaginary,

ie., $\pm iq$ $(p = 0)$ then $y = c_1 \cos qx + c_2 \sin qx$

Note : *If the complex root pair* $p \pm iq$ *is repeated n times (the d.e is of order* $2n$*) then the general solution is given by*

$$y = e^{px} [(c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1}) \cos qx$$

$$+ (c_1{}' + c_2{}' x + c_3{}' x^2 + \cdot + c_n{}' x^{n-1}) \sin qx]$$

Observe the following illustrative table for writing the complimentary function based on the roots of the auxilary equation.

Roots of the A.E.	**Complimentary function** (y_c)
1. 2, 3	$c_1 e^{2x} + c_2 e^{3x}$
2. 1, −1, 2, −2	$c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$
3. 3, 3	$(c_1 + c_2 x) e^{3x}$
4. 0, −2, −2, −2	$c_1 e^{0x} + (c_2 + c_3 x + c_4 x^2) e^{-2x}$
5. ± 2*i*	$c_1 \cos 2x + c_2 \sin 2x$
6. 3 ± 2*i*	$e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$
7. 1, 1, 1 ± *i*	$(c_1 + c_2 x) e^{x} + e^{x}(c_3 \cos x + c_4 \sin x)$
8. 1 ± 2*i*, 1 ± 2*i*	$e^{x} \{(c_1 + c_2 x) \cos 2x + (c_3 + c_4 x) \sin 2x\}$

Further we can as well write down the D.E based on the roots of A.E. With reference to the roots in the illustrative table, the associated D.Es are as follows :

1. $(D-2)(D-3)y = 0$ or $(D^2-5D+6)y = 0$
2. $(D-1)(D+1)(D-2)(D+2)y = 0$ or $(D^4-5D^2+4)y = 0$
3. $(D-3)^2 = 0$ or $(D^2-6D+9)y = 0$
4. $D(D+2)^3 y = 0$
5. $(D-2i)(D+2i)y = 0$ or $(D^2+4)y = 0$
6. $(\overline{D-3}-2i)(\overline{D-3}+2i)y = 0$ or $(D-3)^2+4 = 0$ or $(D^2-6D+13)y = 0$
7. $(D-1)^2(\overline{D-1}-i)(\overline{D-1}+i)y = 0$ or $(D-1)^2[(D-1)^2+1]y = 0$ or $(D-1)^2(D^2-2D+2)y = 0$
8. $[(\overline{D-1}-2i)(\overline{D-1}+2i)]^2 y = 0$ or $[(D-1)^2+4]^2 y = 0$ or $(D^2-2D+5)^2 y = 0$

Working procedure for problems to solve a homogeneous differential equation with constant coefficients

- The given d.e is put in the form $f(D)y = 0$
- Form the A.E $f(m) = 0$ and solve the same *(Here we need to adopt various techniques to solve the equation including the synthetic division method.)*
- Based on the nature of the roots of the A.E we write the C.F which itself is the general solution of the d.e taking into account the dependent and the independent variables involved in the d.e.

WORKED PROBLEMS

1. *Solve :* $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 8y = 0$

\>> We have $(D^3-2D^2+4D-8)y = 0$ where $D = \dfrac{d}{dx}$

A.E is $m^3-2m^2+4m-8 = 0$

(In the case of a cubic equation we first look out for a possible factorization)

ie., $m^2(m-2)+4(m-2) = 0$

ie., $(m-2)(m^2+4) = 0 \Rightarrow m = 2,\ m = \sqrt{-4} = \sqrt{4i^2} = \pm 2i$

Roots of the A.E are $2, \pm 2i$

Thus $y = c_1 e^{2x} + c_2\cos 2x + c_3\sin 2x$ is the general solution.

2. *Solve :* $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} + 6y = 0$

>> We have $(D^3 + 6D^2 + 11D + 6)\, y = 0$

A.E is $m^3 + 6m^2 + 11m + 6 = 0$ *(This cannot be factorized)*

We shall find one root by inspection by taking values for $m = 1, -1, 2, -2$ etc. Since all the terms are positive we have to try only negative values for m. Putting $m = -1$ we have

$(-1)^3 + 6(-1)^2 + 11(-1) + 6 = -1 + 6 - 11 + 6 = 0$

$\therefore\ m = -1$ is a root. The other roots can be found through the process of synthetic division.

$$\begin{array}{r|rrrrl} -1 & 1 & 6 & 11 & 6 & \rightarrow \text{all the coefficients are written.} \\ & 0 & -1 & -5 & -6 & \rightarrow \text{zero to be written below the first coefficient \& added.} \\ \hline & 1 & 5 & 6 & \boxed{0} & \rightarrow \text{resultant to be multiplied with } -1 \text{ and added.} \end{array}$$

Since the last figure is zero we respectively attach m^2, m to the first two coefficients. That is, $m^2 + 5m + 6 = 0$.

Solving : $(m+2)(m+3) = 0 \ \therefore\ m = -2, -3$

Hence $m = -1, -2, -3$ are the roots of the A.E which are all real and distinct.

Thus $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$ is the general solution.

---

3. *Solve :* $(D^3 - 3D + 2)\, y = 0$

>> A.E is $m^3 - 3m + 2 = 0$. $m = 1$ is a root by inspection. The other two roots can be found through synthetic division.

$$\begin{array}{r|rrrr} 1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

Now, $m^2 + m - 2 = 0$ or

$(m+2)(m-1) = 0 \Rightarrow m = -2, 1$

Hence $m = 1, 1, -2$ are the roots of the A.E

Thus $y = (c_1 + c_2 x)\, e^x + c_3 e^{-2x}$ is the general solution.

---

4. *Solve :* $4y''' + 4y'' + y' = 0$

>> We have $(4D^3 + 4D^2 + D)\,y = 0$

A.E is $4m^3 + 4m^2 + m = 0$ or $m\,(4m^2 + 4m + 1) = 0$

ie., $m\,(2m+1)^2 = 0 \Rightarrow m = 0,\ m = -1/2,\ -1/2$

Thus $y = c_1 + (c_2 + c_3\,x)\,e^{-x/2}$ is the general solution.

---

5. *Solve :* $\dfrac{d^3y}{dx^3} + y = 0$

>> We have $(D^3 + 1)\,y = 0$

A.E is $m^3 + 1 = 0,\ m = -1$ is a root by inspection. The other two roots are found by synthetic division.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & 0 & 1 \\ & 0 & -1 & 1 & -1 \\ \hline & 1 & -1 & 1 & \boxed{0} \end{array}$$

Now let us solve $m^2 - m + 1 = 0$

$$\therefore \quad m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3i^2}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Hence $m = -1,\ 1/2 \pm i\,(\sqrt{3}/2)$ are the roots of the A.E.

Thus $y = c_1\,e^{-x} + e^{x/2}\left\{ c_2 \cos(\sqrt{3}/2)\,x + c_3 \sin(\sqrt{3}/2)\,x \right\}$

---

6. *Solve :* $(4D^4 - 4D^3 - 23D^2 + 12D + 36)\,y = 0$

>> A.E is $4m^4 - 4m^3 - 23m^2 + 12m + 36 = 0$

If $m = 2$: $64 - 32 - 92 + 24 + 36 = 124 - 124 = 0$

$\Rightarrow m = 2$ is a root by inspection.

Now by synthetic division,

$$\begin{array}{r|rrrrr} 2 & 4 & -4 & -23 & 12 & 36 \\ & 0 & 8 & 8 & -30 & -36 \\ \hline & 4 & 4 & -15 & -18 & \boxed{0} \end{array}$$

Now, $4m^3 + 4m^2 - 15m - 18 = 0$

If $m = 2,\ 32 + 16 - 30 - 18 = 48 - 48 = 0$

Again $m = 2$ is a root. By synthetic division,

$$\begin{array}{r|rrrr} 2 & 4 & 4 & -15 & -18 \\ & 0 & 8 & 24 & 18 \\ \hline & 4 & 12 & 9 & \boxed{0} \end{array}$$

Now, $4m^2 + 12m + 9 = 0$

or $(2m + 3)^2 = 0 \Rightarrow m = -3/2, -3/2$

Hence roots of the A.E are $2, 2\ ;\ -3/2, -3/2$

Thus $y = (c_1 + c_2 x) e^{2x} + (c_3 + c_4 x) e^{-3x/2}$ is the general solution.

7. *Solve :* $(D^4 - 5D^2 + 4) y = 0$

\>> A.E is $m^4 - 5m^2 + 4 = 0$

or $(m^2 - 1)(m^2 - 4) = 0$

or $(m - 1)(m + 1)(m - 2)(m + 2) = 0$

$\therefore$ roots of the A.E are $1, -1, 2, -2$

Thus $y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$ is the general solution.

8. *Solve :* $\dfrac{d^4 y}{dt^4} + 8\dfrac{d^2 y}{dt^2} + 16y = 0$

\>> We have, $(D^4 + 8D^2 + 16) y = 0$, where $D = \dfrac{d}{dt}$

A.E is $m^4 + 8m^2 + 16 = 0$ or $(m^2 + 4)^2 = 0$

$\therefore$ roots of the A.E are $\pm 2i, \pm 2i$ *(repeated imaginary roots)*
(Take a note that 't' is the independent variable)

Thus $y = (c_1 + c_2 t) \cos 2t + (c_3 + c_4 t) \sin 2t$ is the general solution.

9. *Solve :* $(D^4 + 64) y = 0$

\>> A.E is $m^4 + 64 = 0$. *ie.,* $(m^2)^2 + (8)^2 = 0$

Using $a^2 + b^2 = (a + b)^2 - 2ab$ we have

$$(m^2)^2 + (8)^2 = (m^2 + 8)^2 - 16m^2 = (m^2 + 8)^2 - (4m)^2$$

ie., $m^4 + 64 = \{(m^2 + 8) - 4m\}\{(m^2 + 8) + 4m\}$

Hence we have, $m^2 - 4m + 8 = 0$ and $m^2 + 4m + 8 = 0$

$$\therefore \quad m = \frac{4 \pm \sqrt{16 - 32}}{2} \quad ; \; m = \frac{-4 \pm \sqrt{16-32}}{2}$$

$$m = \frac{4 \pm 4i}{2} = 2 \pm 2i \quad ; \; m = \frac{-4 \pm 4i}{2} = -2 \pm 2i$$

$\therefore$ roots of the A.E are $2 \pm 2i, \; -2 \pm 2i$ (*complex roots*)

Thus $y = e^{2x} (c_1 \cos 2x + c_2 \sin 2x) + e^{-2x} (c_3 \cos 2x + c_4 \sin 2x)$

---

10. *Solve :* $\dfrac{d^4 x}{dt^4} - 2 \dfrac{d^3 x}{dt^3} + \dfrac{d^2 x}{dt^2} = 0$

>> We have $(D^4 - 2D^3 + D^2) x = 0$ where $D = \dfrac{d}{dt}$

A.E is $m^4 - 2m^3 + m^2 = 0$

ie., $m^2 (m^2 - 2m + 1) = 0$ or $m^2 (m - 1)^2 = 0$

$\therefore$ roots of the A.E are $0, 0 \; ; \; 1, 1$. [*Note that* $x = x (t)$]

Thus $x = (c_1 + c_2 t) + (c_3 + c_4 t) e^t$ is the general solution.

---

11. *Solve :* $(4D^4 - 8D^3 - 7D^2 + 11D + 6) y = 0$

>> A.E is $4m^4 - 8m^3 - 7m^2 + 11m + 6 = 0$

If $m = -1$: $4 + 8 - 7 - 11 + 6 = 18 - 18 = 0$

$\therefore \; m = -1$ is a root by inspection.

-1	4	-8	-7	11	6
	0	-4	12	-5	-6
	4	-12	5	6	0

Now, $4m^3 - 12m^2 + 5m + 6 = 0$

If $m = 2$: $32 - 48 + 10 + 6 = 48 - 48 = 0 \;\; \therefore \; m = 2$ is also a root.

2	4	-12	5	6
	0	8	-8	-6
	4	-4	-3	0

Now, $4m^2 - 4m - 3 = 0$

$\therefore$ $4m^2 - 6m + 2m - 3 = 0$

ie., $2m(2m-3) + 1(2m-3) = 0$ or $(2m+1)(2m-3) = 0$

$\Rightarrow$ $m = -1/2,\ 3/2$

Hence the roots of the A.E are $-1,\ 2,\ -1/2,\ 3/2$

Thus $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-x/2} + c_4 e^{3x/2}$ is the general solution.

---

12. *Solve :* $(D^5 - D^4 - D + 1)y = 0$

>> A.E is $m^5 - m^4 - m + 1 = 0$

ie., $m^4(m-1) - 1(m-1) = 0$

ie., $(m-1)(m^4-1) = 0$ or $(m-1)(m^2-1)(m^2+1) = 0$

ie., $(m-1)(m-1)(m+1)(m^2+1) = 0$

$\therefore$ the roots of the A.E are $1,\ 1,\ -1,\ \pm i$

Thus $y = (c_1 + c_2 x)e^x + c_3 e^{-x} + (c_4 \cos x + c_5 \sin x)$

---

13. *If* $\dfrac{d^4 x}{dt^4} = m^4 x$ *show that* $x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$

>> We have $(D^4 - m^4)x = 0$ where $D = \dfrac{d}{dt}$

(Note : *Since the given example involves "m" we should not use the same for writing the A.E)*

A.E is $(p^4 - m^4) = 0$ or $(p^2 + m^2)(p^2 - m^2) = 0$

ie., $p = \pm mi,\ p = \pm m$ are the roots of the A.E ; Also $x = x(t)$

$\therefore$ $x = c_1 \cos mt + c_2 \sin mt + c_3 e^{mt} + c_4 e^{-mt}$ ... (1)

Consider the desired expression :

$$x = c_1 \cos mt + c_2 \sin mt + c_3 \cosh mt + c_4 \sinh mt$$

$$= c_1 \cos mt + c_2 \sin mt + c_3 \left[\frac{e^{mt} + e^{-mt}}{2}\right] + c_4 \left[\frac{e^{mt} - e^{-mt}}{2}\right]$$

$$= c_1 \cos mt + c_2 \sin mt + \left[\frac{c_3 + c_4}{2}\right] e^{mt} + \left[\frac{c_3 - c_4}{2}\right] e^{-mt}$$

Denoting $c_3' = \frac{c_3 + c_4}{2}$ and $c_4' = \frac{c_3 - c_4}{2}$ we get,

$$x = c_1 \cos mt + c_2 \sin mt + c_3' e^{mt} + c_4' e^{-mt} \qquad \ldots (2)$$

We conclude that (1) = (2) since c_3', c_4' are also arbitrary constants.

- -

14. *Solve :* $D^2 (D^2 + 2D)^2 (D^2 + 2D + 2)^3 \; y = 0$

\>> A.E is $m^2 (m^2 + 2m)^2 (m^2 + 2m + 2)^3 = 0$

$\Rightarrow \quad m^2 = 0, \; (m^2 + 2m)^2 = 0, \; (m^2 + 2m + 2)^3 = 0$

$m^2 = 0 \;\therefore\; m = 0, 0$

$(m^2 + 2m)^2 = 0$ or $[m(m+2)]^2 = 0$

ie., $\quad m^2 = 0, \; (m+2)^2 = 0 \quad \therefore \; m = 0, 0, -2, -2$

Also $(m^2 + 2m + 2)^3 = 0$ and let us solve $m^2 + 2m + 2 = 0$

$$m = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$\therefore \quad m = -1 \pm i$ is a triple root.

Hence the roots of the A.E are,

$$m = 0, 0, 0, 0 \; ; \; -2, -2 \; ; \; -1 \pm i, -1 \pm i, -1 \pm i$$

Thus $\quad y = (c_1 + c_2 x + c_3 x^2 + c_4 x^3) + (c_5 + c_6 x) e^{-2x}$

$$+ e^{-x} \left\{ (c_7 + c_8 x + c_9 x^2) \cos x + (c_{10} + c_{11} x + c_{12} x^2) \sin x \right\}$$

- -

Solution of homogeneous differential equation subject to given conidtions

15. *Solve :* $y'' + 4y' + 4y = 0$ *given that* $y = 0, \; y' = -1$ *at* $x = 1$

\>> We have $(D^2 + 4D + 4) y = 0$ where $D = \frac{d}{dx}$

A.E is $m^2 + 4m + 4 = 0$ or $(m+2)^2 = 0 \Rightarrow m = -2, -2$

Hence $\quad y = (c_1 + c_2 x) e^{-2x} \qquad \ldots (1)$

Now $\quad y' = (c_1 + c_2 x)(-2 e^{-2x}) + c_2 e^{-2x} \qquad \ldots (2)$

Consider $y = 0$ at $x = 1$. Hence (1) becomes

$$0 = (c_1 + c_2)e^{-2} \quad \Rightarrow \quad c_1 + c_2 = 0$$

Also $y' = -1$ at $x = 1$. Hence (2) becomes

$$-1 = (c_1 + c_2)(-2e^{-2}) + c_2 e^{-2}. \text{ But } c_1 + c_2 = 0$$

ie., $-1 = c_2(e^{-2})$ or $c_2 = -e^2 \quad \therefore \quad c_1 = -c_2 = e^2$

Substituting these values in (1) we get,

$$y = e^2(1-x)e^{-2x}$$

Thus $\boldsymbol{y = (1-x)e^{2(1-x)}}$ is the particular solution.

---

16. *Solve:* $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$, *given that* $y = 2$ *and* $\dfrac{dy}{dx} = \dfrac{d^2y}{dx^2}$ *when* $x = 0$

\>> We have $(D^2 + 4D + 5)y = 0$

A.E is $m^2 + 4m + 5 = 0$

$$\therefore \quad m = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Hence $y = e^{-2x}(c_1 \cos x + c_2 \sin x)$ ... (1)

$$\frac{dy}{dx} = y' = e^{-2x}(-c_1 \sin x + c_2 \cos x) - 2e^{-2x}(c_1 \cos x + c_2 \sin x)$$

ie., $y' = e^{-2x}(-c_1 \sin x + c_2 \cos x) - 2y$

Also $y'' = e^{-2x}(-c_1 \cos x - c_2 \sin x)$

$$-2e^{-2x}(-c_1 \sin x + c_2 \cos x) - 2y'$$

Consider $y = 2$ when $x = 0$. $(y)_{x=0} = c_1$ from (1) $\therefore$ $2 = c_1$

Consider $y' = y''$ when $x = 0$

$$(y')_{x=0} = c_2 - 2c_1 \; ; \; (y'')_{x=0} = -c_1 - 2c_2 - 2(c_2 - 2c_1)$$

$$\therefore \quad c_2 - 2c_1 = 3c_1 - 4c_2 \text{ or } 5c_1 - 5c_2 = 0 \Rightarrow c_1 = c_2$$

But $c_1 = 2 \quad \therefore \quad c_2 = 2$

Thus by substituting $c_1 = 2 = c_2$ in (1) we have,

$y = 2e^{-2x}(\cos x + \sin x)$ being the particular solution.

17. *Solve the initial value problem :* $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 0$ *with conditions,*

$x(0) = 0, \ \dfrac{dx}{dt}(0) = 1$

>> We have $(D^2 - 3D + 2)x = 0 ; D = \dfrac{d}{dt}$

A.E is given by $m^2 - 3m + 2 = 0$

or $(m-1)(m-2) = 0 \Rightarrow m = 1, 2$

Hence, $x = x(t) = c_1 e^t + c_2 e^{2t}$ ... (1)

Also $x'(t) = \dfrac{dx}{dt} = c_1 e^t + 2c_2 e^{2t}$ ... (2)

Let us consider the conditions $x(0) = 0$ and $x'(0) = 1$. Then (1) and (2) becomes

$0 = c_1 + c_2$ and $1 = c_1 + 2c_2$. By solving we get $c_1 = -1, \ c_2 = 1$

Thus $x = -e^t + e^{2t}$ is the particular solution.

18. *Solve :* $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 5y = 0$ *subject to,* $\dfrac{dy}{dx} = 2, \ y = 1$ *at* $x = 0$

>> We have $(D^2 - 4D + 5)y = 0 \ ; \ D = \dfrac{d}{dx}$

A.E is given by $m^2 - 4m + 5 = 0$

$$m = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Hence, $y = e^{2x}(c_1 \cos x + c_2 \sin x)$ ... (1)

Also $\dfrac{dy}{dx} = e^{2x}(-c_1 \sin x + c_2 \cos x) + 2e^{2x}(c_1 \cos x + c_2 \sin x)$ ... (2)

By the condition, $y = 1$ at $x = 0$, (1) becomes

$1 = 1(c_1 + 0) \quad \therefore \ c_1 = 1$

Also by the condition $\frac{dy}{dx} = 2$ at $x = 0$, (2) becomes

$2 = c_2 + 2c_1$. Using $c_1 = 1$ we get $c_2 = 0$

Thus $y = e^{2x} \cos x$ is the particular solution.

---

19. *Solve the initial value problem* $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 29x = 0$, *given that* $x(0) = 0,\ \frac{dx}{dt}(0) = 15$

>> We have $(D^2 + 4D + 29)x = 0$, where $D = \frac{d}{dt}$

A.E is given by $m^2 + 4m + 29 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 116}}{2} = \frac{-4 \pm 10i}{2} = -2 \pm 5i$$

Hence, $x(t) = e^{-2t}(c_1 \cos 5t + c_2 \sin 5t)$ . . . (1)

$\frac{dx}{dt} = x'(t) = e^{-2t}(-5c_1 \sin 5t + 5c_2 \cos 5t) - 2e^{-2t}(c_1 \cos 5t + c_2 \sin 5t)$ . . . (2)

Let us consider $x(0) = 0$ and $x'(0) = 15$

(1) and (2) respectively becomes $0 = c_1$ and $15 = 5c_2 - 2c_1$

$\therefore$ $c_1 = 0$ and $c_2 = 3$

Thus $x(t) = 3e^{-2t} \sin 5t$ is the required particular solution

---

20. *Solve* : $x''(t) + w^2 x(t) = 0$, *subject to* : $x = a,\ x' = aw$ *when* $t = 2\pi/w$

We have $(D^2 + w^2)x(t) = 0$, where $D = \frac{d}{dt}$

A.E is $m^2 + w^2 = 0 \Rightarrow m = \pm iw$

Hence, $x(t) = c_1 \cos wt + c_2 \sin wt$ . . . (1)

Also, $x'(t) = -w c_1 \sin wt + w c_2 \cos wt$

By data, $x(2\pi/w) = a$ and $x'(2\pi/w) = aw$

Hence (1) and (2) respectively give us,

$a = c_1$ and $aw = wc_2$ or $a = c_2$

Thus by substituting $c_1 = a = c_2$ in (1) we get,

$x(t) = a(\cos wt + \sin wt)$, being the particular solution.

EXERCISES

Solve the following differential equations :

1. $\dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} - 6y = 0$
2. $\dfrac{d^3y}{dx^3} - 8y = 0$
3. $(D^3 - 3D^2 + 4)x = 0$, where $D = \dfrac{d}{dt}$
4. $16y''' - 8y'' + y' = 0$
5. $2x'''(t) + 5x''(t) - 12x'(t) = 0$
6. $(D^4 - 2D^3 + 2D^2 - 2D + 1)y = 0$
7. $\dfrac{d^4z}{dt^4} - 16z = 0$
8. $(D^4 - 4D^3 + 8D^2 - 8D + 4)x(t) = 0$
9. $\dfrac{d^2x}{dt^2} + \mu x = 0$, $(\mu > 0)$ given that $x = a$, $\dfrac{dx}{dt} = 0$ when $t = \pi/\sqrt{\mu}$
10. $(D^3 + D^2 - 2)y = 0$ subject to the conditions $y(0) = 2$, $y'(0) = 2$, $y''(0) = -3$

ANSWERS

1. $y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$
2. $y = c_1 e^{2x} + e^{-x}(c_2 \cos\sqrt{3}\,x + c_3 \sin\sqrt{3}\,x)$
3. $x = c_1 e^{-t} + (c_2 + c_3 t)e^{2t}$
4. $y = c_1 + (c_2 + c_3 x)e^{x/4}$
5. $x = c_1 + c_2 e^{-4t} + c_3 e^{3t/2}$
6. $y = (c_1 + c_2 x)e^x + (c_3 \cos x + c_4 \sin x)$
7. $z = c_1 e^{2t} + c_2 e^{-2t} + c_3 \cos 2t + c_4 \sin 2t$
8. $x = e^t(c_1 + c_2 t)\cos t + (c_3 + c_4 t)\sin t$
9. $x = -a\cos\sqrt{\mu}\,t$
10. $y = e^x + e^{-x}\ (\cos x + 2\sin x)$

2.3 Inverse differential operator and the particular integral

We have the differential operator $D = \frac{d}{dx}$

Let us suppose that $D[G(x)] = F(x)$

Then $\int F(x)\,dx = G(x)$ is an inverse operation.

Equivalently we write $\frac{1}{D}F(x) = G(x)$

The operator $\frac{1}{D}$ that stands for the integral is called an *inverse differential operator.*
$\frac{1}{D^2}, \frac{1}{D^3}$ etc. stands for successive integration.

Examples : $\frac{1}{D}x^2 = \int x^2\,dx = \frac{x^3}{3}$; $\frac{1}{D}\cos x = \sin x$

$$\frac{1}{D^2}e^{3x} = \int\int e^{3x}\,dx\,dx = \int\frac{e^{3x}}{3}\,dx = \frac{e^{3x}}{9}$$

Note : *We omit the constant of integration in order to have a specific expression.*

Next, if 'a' is a constant let us find $\frac{1}{D-a}F(x)$

Suppose $\frac{1}{D-a}F(x) = y$ then $(D-a)y = F(x)$ or $\frac{dy}{dx} - ay = F(x)$ which is a linear differential equation whose solution is given by

$$ye^{-ax} = \int F(x)e^{-ax}dx \quad \text{or} \quad y = e^{ax}\int F(x)e^{-ax}dx$$

$$\frac{1}{D-a}F(x) = e^{ax}\int F(x)e^{-ax}dx \qquad \ldots(1)$$

Also $$\frac{1}{(D-b)(D-a)}F(x) = \frac{1}{D-b}\left\{\frac{1}{D-a}F(x)\right\} \qquad \ldots(2)$$

This is nothing but the repeated application of the result (1).

Note : *Alternatively* $\frac{1}{(D-b)(D-a)}$ *can be expressed in the form* $\frac{A}{D-b}+\frac{B}{D-a}$ *by partial fractions so that*
$\frac{1}{(D-b)(D-a)}F(x) = A\left\{\frac{1}{D-b}F(x)\right\}+B\left\{\frac{1}{D-a}F(x)\right\}$ *and (1) can be applied independently.*

Suppose $\frac{1}{(D-b)(D-a)} F(x) = y$ then $(D-b)(D-a)y = F(x)$

In general we can say that,

$$f(D)y = F(x) \Rightarrow y = \frac{F(x)}{f(D)}$$

But we have already said that if $f(D)y = F(x)$ then $y = y_p(x)$ free from the arbitrary constants is called the particular solution or particular integral (P.I) denoted by y_p. Thus we can say that

$$\boldsymbol{P \cdot I = y_p = \frac{F(x)}{f(D)}}$$

in respect of the equation in the form $f(D)y = F(x)$

Further it may also be noted that if $F(x) = aF_1(x) + bF_2(x)$, where a and b are constants, then

$$P \cdot I = y_p = \frac{F(x)}{f(D)} = a \cdot \frac{F_1(x)}{f(D)} + b \cdot \frac{F_2(x)}{f(D)}$$

We present a few illustrations of finding P.I by this basic method .

(i) Let us find $\frac{1}{D-1} \cos 2x$

Using $\frac{1}{D-a} F(x) = e^{ax} \int e^{-ax} F(x)\, dx$ we have

$$\frac{1}{D-1} \cos 2x = e^{x} \int e^{-x} \cos 2x\, dx \qquad \ldots (1)$$

Using the standard integral,

$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$ in the R.H.S of (1) we get,

$$\frac{1}{D-1} \cos 2x = e^{x} \cdot \frac{e^{-x}}{5} (-\cos 2x + 2 \sin 2x)$$

ie., $$\frac{1}{D-1} \cos 2x = \frac{1}{5} (2 \sin 2x - \cos 2x)$$

(ii) Let us find $\frac{1}{D^2+3D+2} e^{x}$

Here $\frac{1}{D^2+3D+2} = \frac{1}{(D+1)(D+2)} = \frac{1}{D+1} - \frac{1}{D+2}$ by partial fractions.

$$\therefore \quad \frac{1}{D^2+3D+2}e^x = \frac{1}{D+1}e^x - \frac{1}{D+2}e^x$$

$$= e^{-x}\int e^x \cdot e^x\,dx - e^{-2x}\int e^{2x}\cdot e^x\,dx$$

$$= e^{-x}\cdot\frac{e^{2x}}{2} - e^{-2x}\cdot\frac{e^{3x}}{3} = \frac{e^x}{2} - \frac{e^x}{3} = \frac{e^x}{6}$$

Thus $\quad \dfrac{1}{D^2+3D+2}e^x = \dfrac{e^x}{6}$

This fundamental method becomes highly tedious when $f(D)$ is in higher orders and $F(x)$ is in a complicated form as we have to necessarily decompose $f(D)$ into the form $(D-m_1)(D-m_2)\cdots(D-m_n)$ for applying the result. Hence we proceed to illustrate simple and shortcut methods for finding the P.I when $F(x)$ in $f(D)y = F(x)$ belong to some specific form.

We discuss five specific forms of $F(x)$

2.4 Specific forms of the particular integral

2.41 Type-1 : P.I of the form $\dfrac{e^{ax}}{f(D)}$

We have to obtain the particular solution of the d.e $f(D)y = e^{ax}$

Let $\quad f(D) = D^2 + a_1 D + a_2$

We have $\quad D(e^{ax}) = ae^{ax},\ D^2(e^{ax}) = a^2 e^{ax}$

$$\therefore \quad f(D)e^{ax} = (D^2+a_1 D+a_2)e^{ax} = D^2(e^{ax}) + a_1 D(e^{ax}) + a_2 e^{ax}$$

$$= a^2 e^{ax} + a_1\cdot a\,e^{ax} + a_2 e^{ax}$$

$$= (a^2 + a_1 a + a_2)e^{ax} = f(a)e^{ax}$$

Thus $\quad f(D)e^{ax} = f(a)e^{ax}$

Operating with $1/f(D)$ on both sides we get,

$$e^{ax} = f(a)\cdot\frac{1}{f(D)}e^{ax} \quad \text{or} \quad \frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)}$$

Thus $\dfrac{e^{ax}}{f(D)} = \dfrac{e^{ax}}{f(a)}$ *(D being replaced by a) where* $f(a) \neq 0$

Suppose $f(a) = 0$ then $(D-a)$ is a factor of $f(D)$

ie., $f(D) = (D-a)\phi(D)$ ...(1)

$$\frac{e^{ax}}{f(D)} = \frac{e^{ax}}{(D-a)\phi(D)} = \frac{1}{D-a} \cdot \frac{e^{ax}}{\phi(a)}$$

$$= \frac{1}{\phi(a)} \left\{ \frac{1}{(D-a)} e^{ax} \right\} = \frac{1}{\phi(a)} e^{ax} \int e^{-ax} e^{ax}\, dx$$

ie., $$\frac{e^{ax}}{f(D)} = \frac{1}{\phi(a)} e^{ax} \int 1\, dx = \frac{1}{\phi(a)} x\, e^{ax} \qquad \ldots(2)$$

From (1) $f'(D) = (D-a)\phi'(D) + \phi(D)$ by treating D as a variable.

$\therefore \quad f'(a) = 0 + \phi(a)$ or $f'(a) = \phi(a)$

Thus (2) becomes

$$\frac{e^{ax}}{f(D)} = x \cdot \frac{e^{ax}}{f'(a)} \text{ where } f(a) = 0 \text{ and } f'(a) \neq 0$$

This result can be extended further also.

That is, if $f'(a) = 0$, $\dfrac{e^{ax}}{f(D)} = x^2 \cdot \dfrac{e^{ax}}{f''(a)}$ and so on.

Note : *The result also holds good for the following functions.*

(i) e^{ax+b} *(ii) any constant* k *as we can write* $k = k \cdot e^{0x}$

(iii) $a^x = (e^{\log a})^x = e^{bx}$ *where* $b = \log a$

(iv) $\sinh ax = 1/2 \cdot (e^{ax} - e^{-ax})$ *and* $\cosh ax = 1/2 \cdot (e^{ax} + e^{-ax})$

Illustrations

1. $$\frac{e^{2x}}{D+3} = \frac{e^{2x}}{2+3} = \frac{e^{2x}}{5}$$

2. $$\frac{e^x}{D^2+3D+2} = \frac{e^x}{1^2+3 \cdot 1+2} = \frac{e^x}{6}$$

[Refer the previous article, as we have found P.I by the basic method]

3. $$\frac{e^{3x+4}}{D^3+D^2+D+1} = \frac{e^{3x+4}}{(3)^3+(3)^2+3+1} = \frac{e^{3x+4}}{40}$$

4. $$\frac{4^x}{D^2+5D+6} = \frac{(e^{\log 4})^x}{D^2+5D+6} = \frac{e^{\log 4 \cdot x}}{D^2+5D+6}$$

$$= \frac{e^{\log 4 \cdot x}}{(\log 4)^2+5(\log 4)+6} = \frac{4^x}{(\log 4)^2+5\log 4+6}$$

5. $$\frac{e^{-x}}{D^2+7D+6} = \frac{e^{-x}}{(-1)^2+7(-1)+6} = \frac{e^{-x}}{1-7+6} \quad (Dr. = 0)$$

$$= x \cdot \frac{e^{-x}}{2D+7} = x \cdot \frac{e^{-x}}{2(-1)+7} = \frac{x\,e^{-x}}{5}$$

6. $$\frac{e^{-2x}}{D^2+4D+4} = \frac{e^{-2x}}{(-2)^2+4(-2)+4} = \frac{e^{-2x}}{4-8+4} \quad (Dr. = 0)$$

$$= x \cdot \frac{e^{-2x}}{2D+4} = x \cdot \frac{e^{-2x}}{-4+4} \quad (Again\ Dr. = 0)$$

$$= x^2\,\frac{e^{-2x}}{2} \quad ie., \quad \frac{e^{-2x}}{D^2+4D+4} = \frac{x^2 e^{-2x}}{2}$$

7. $$\frac{\cosh 2x}{D^2-4} = \frac{\frac{1}{2}(e^{2x}+e^{-2x})}{D^2-4} = \frac{1}{2}\,\frac{e^{2x}}{D^2-4} + \frac{1}{2}\,\frac{e^{-2x}}{D^2-4} = p_1 + p_2$$

$$p_1 = \frac{1}{2}\,\frac{e^{2x}}{D^2-4} = \frac{1}{2}\,\frac{e^{2x}}{2^2-4} \quad (Dr. = 0)$$

$$= \frac{1}{2} \cdot x\,\frac{e^{2x}}{2D} = \frac{1}{2} \cdot x \cdot \frac{e^{2x}}{2 \cdot 2} = \frac{x\,e^{2x}}{8}$$

$$p_2 = \frac{1}{2}\,\frac{e^{-2x}}{D^2-4} = \frac{1}{2}\,\frac{e^{-2x}}{(-2)^2-4} \quad (Dr. = 0)$$

$$= \frac{1}{2}\,x\,\frac{e^{-2x}}{2D} = \frac{1}{2}\,x \cdot \frac{e^{-2x}}{2(-2)} = \frac{x\,e^{-2x}}{-8}$$

$$\therefore \quad P.I = p_1 + p_2 = \frac{x\,e^{2x}}{8} + \frac{x\,e^{-2x}}{-8} = \frac{x}{4}\left(\frac{e^{2x}-e^{-2x}}{2}\right) = \frac{x \sinh 2x}{4}$$

2.42 Type-2 : P.I of the form $\frac{\sin ax}{f(D)}, \frac{\cos ax}{f(D)}$

$$D(\sin ax) = a\cos ax,\ D^2(\sin ax) = -a^2 \sin ax$$

$$D^3(\sin ax) = -a^3 \cos ax,\ D^4(\sin ax) = a^4 \sin ax$$

We observe that $(D^2)^1 \sin ax = (-a^2)^1 \sin ax,$

$$(D^2)^2 \sin ax = (-a^2)^2 \sin ax \text{ etc } (D^2)^n \sin ax = (-a^2)^n \sin ax$$

Hence, $f(D^2)\sin ax = f(-a^2)\sin ax$

Operating with $1/f(D^2)$ on both sides we have,

$$\sin ax = f(-a^2)\cdot\frac{1}{f(D^2)}\sin ax \text{ or } \frac{\sin ax}{f(D^2)} = \frac{\sin ax}{f(-a^2)} \text{ where } f(-a^2) \neq 0.$$

(D^2 *being replaced by* $-a^2$)

It can be easily seen that the working does hold good for $\cos ax$ also.

$$\therefore \quad \frac{\cos ax}{f(D^2)} = \frac{\cos ax}{f(-a^2)} \text{ where } f(-a^2) \neq 0$$

However if $f(-a^2) = 0$, as in the previous type we can show that

$$\frac{\sin ax}{f(D^2)} = x\cdot\frac{\sin ax}{f'(-a^2)}\ ;\ \frac{\cos ax}{f(D^2)} = x\cdot\frac{\cos ax}{f'(-a^2)} \text{ etc.}$$

Note : (1) *The results also hold good for the functions* $\sin(ax+b)$ *and* $\cos(ax+b)$

(2) *Functions like* $\sin ax \cos bx,\ \sin ax \sin bx,\ \cos ax \cos bx$ *have to be converted into a sum by using basic trigonometric formulae.*

(3) *After replacing* D^2 *by* $-a^2$, $f(D)$ *transforms into the form* $\alpha D \pm \beta$ *(*α *&* β *are constants) in which case we multiply both the numerator and denominator by* $\alpha D \mp \beta$ *for proceeding further in obtaining the P.I. However if* $f(D)$ *involves only even powers of D then, the same transforms into a constant on replacing* D^2 *by* $-a^2$

Illustrations

1. $\frac{\cos 2x}{D-1}$ We need to multiply and divide by $(D+1)$

$$P.I = \frac{(D+1)\cos 2x}{(D+1)(D-1)} = \frac{D\cos 2x + \cos 2x}{D^2-1} = \frac{-2\sin 2x + \cos 2x}{D^2-1}.$$

Here $a = 2$ and we replace D^2 by -4

$$P.I = \frac{-2\sin 2x + \cos 2x}{-4-1} = \frac{1}{5}(2\sin 2x - \cos 2x)$$

[*Refer (i) in the article 2.3 as we have found P.I by the basic method*]

2. $\dfrac{\sin 3x}{D^2+4D+3}$ Here $a = 3$, replace D^2 by -9

$$P.I = \frac{\sin 3x}{-9+4D+3} = \frac{\sin 3x}{4D-6} \quad \text{Multiply and divide by } 4D+6.$$

$$= \frac{(4D+6)\sin 3x}{(4D+6)(4D-6)} = \frac{4(3\cos 3x)+6\sin 3x}{16D^2-36}$$

$$P.I = \frac{6(2\cos 3x+\sin 3x)}{16(-9)-36} = \frac{6(2\cos 3x+\sin 3x)}{-180} = \frac{-1}{30}(2\cos 3x+\sin 3x)$$

2.43 Type-3 : P.I of the form $\frac{\phi(x)}{f(D)}$ where $\phi(x)$ is a polynomial in x

We are seeking the particular solution of $f(D)y = \phi(x)$ where

$$\phi(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n$$

It is evident that $y = y_p$ will also be a polynomial in x. Hence P.I is found by division. By writing $\phi(x)$ in descending powers of x and $f(D)$ in ascending powers of D, the division gets completed without any remainder. The quotient so obtained in the process of division will be the particular integral.

Illustrations

1. $\dfrac{2x^2+4x+1}{D^2+3D+2}$

$$P\cdot I = \frac{2x^2+4x+1}{2+3D+D^2} \quad \text{and we shall divide.}$$

$$\begin{array}{r|l}
 & x^2 - x + 1 \\
\hline
2+3D+D^2 & 2x^2 + 4x + 1 \\
 & 2x^2 + 6x + 2 \\
 & (-) \quad (-) \quad (-) \\
\hline
 & -2x - 1 \\
 & -2x - 3 \\
 & (+) \quad (+) \\
\hline
 & +2 \\
 & 2 \\
 & (-) \\
\hline
 & 0
\end{array}$$

Note : $3D(x^2) = 6x$; $D^2(x^2) = 2$

Note : $3D(-x) = -3$, $D^2(-x) = 0$

Thus $P \cdot I = x^2 - x + 1$

Remark : *It can be easily seen that,* $(D^2 + 3D + 2)(x^2 - x + 1) = 2x^2 + 4x + 1$

2. $\dfrac{x^2 - 3x + 1}{D^3 - D^2}$; $P \cdot I = \dfrac{x^2 - 3x + 1}{-D^2 + D^3}$ for division.

$$\begin{array}{r|l} & -x^4/12 + x^3/6 \\ \hline -D^2 + D^3 & x^2 - 3x + 1 \\ & x^2 - 2x \\ & (-) \quad (+) \\ \hline & -x + 1 \\ & -x + 1 \\ & (+) \quad (-) \\ \hline & 0 \end{array}$$

First we need

$$\frac{x^2}{-D^2} = -\iint x^2 \, dx \, dx = \frac{-x^4}{12}$$

Also $D^3(-x^4/12) = -2x$

Now we need

$$\frac{-x}{-D^2} = \iint x \, dx \, dx = \frac{x^3}{6}$$

Also $D^3(x^3/6) = 1$

Thus $P.I = \dfrac{x^3}{6} - \dfrac{x^4}{12} = \dfrac{x^3}{12}(2 - x)$

$\boxed{2.44}$ Type-4 : P.I of the form $\dfrac{e^{ax} V}{f(D)}$ where V is a function of x

Let W be any function of x.

$\therefore \quad D(e^{ax} W) = e^{ax} DW + a e^{ax} W = e^{ax}(D + a) W$

Operating with D again we have,

$$\begin{aligned} D^2(e^{ax} W) &= e^{ax} D(D + a) W + D(e^{ax})(D + a) W \\ &= e^{ax}(D^2 + aD) W + a e^{ax}(DW + aW) \\ &= e^{ax}(D^2 W + 2a DW + a^2 W) \end{aligned}$$

ie., $D^2(e^{ax} W) = e^{ax}(D + a)^2 W$ and so on.

$\therefore \quad f(D)(e^{ax} W) = e^{ax} f(D + a) W \qquad \ldots (1)$

Let $f(D + a) W = V$. Hence $W = \dfrac{1}{f(D + a)} V$

Thus (1) becomes

$$f(D) e^{ax} \frac{1}{f(D + a)} V = e^{ax} V$$

Now operating with $1/f(D)$ on both sides we have,

$$e^{ax}\frac{1}{f(D+a)}V=\frac{e^{ax}V}{f(D)}$$

Thus $$\frac{e^{ax}V}{f(D)}=e^{ax}\cdot\frac{V}{f(D+a)}$$

Remark : *If e^{ax} is multiplied with some function of x, we need to first shift D to $D+a$. Then the computation of P.I will follow the type 2 or 3.*

Illustrations

1. $$\frac{e^{2x}\sin 3x}{D^2-1}$$

Shifting D to $(D+2)$ we have,

$$e^{2x}\frac{\sin 3x}{(D+2)^2-1}=e^{2x}\cdot\frac{\sin 3x}{D^2+4D+3}$$

We have to find $\frac{\sin 3x}{D^2+4D+3}$ which is same as illustration-2 in Type-2

Thus $$\frac{e^{2x}\sin 3x}{D^2-1}=e^{2x}\cdot\frac{-1}{30}(2\cos 3x+\sin 3x)=\frac{-e^{2x}}{30}\cdot(2\cos 3x+\sin 3x)$$

2. $$\frac{e^{-x}(2x^2+4x+1)}{D^2+5D+6}$$

Shifting D to $(D-1)$ we have

$$e^{-x}\cdot\frac{2x^2+4x+1}{(D-1)^2+5(D-1)+6}=e^{-x}\cdot\frac{2x^2+4x+1}{D^2+3D+2}$$

P.I of $\frac{2x^2+4x+1}{D^2+3D+2}$ is same as illustration (1) of Type-3.

Thus $$\frac{e^{-x}(2x^2+4x+1)}{(D^2+5D+6)}=e^{-x}(x^2-x+1)$$

$\boxed{2.45}$ Type-5 : P.I of the form $\frac{xV}{f(D)}$, $\frac{x^nV}{f(D)}$ where V is a function of x

Let W be any function of x.

$$D^n(xW)=xD^nW+n\cdot 1\cdot D^{n-1}W\text{, by Leibnitz theorem.}$$

We shall write nD^{n-1} present in the R.H.S equal to $\frac{d}{dD}(D^n)$ by regarding D as a variable.

$$\therefore \quad D^n(xW) = x(D^n W) + \frac{d}{dD}(D^n)W$$

Hence $\quad f(D)xW = x \cdot f(D)W + f'(D)W \qquad \ldots(1)$

Now, let $f(D)W = V$ and hence $W = \frac{1}{f(D)}V$

$\therefore$ (1) becomes

$$f(D)x\frac{1}{f(D)}V = x \cdot V + f'(D)\frac{1}{f(D)}V$$

Operating with $1/f(D)$ on both sides we have

$$x\frac{1}{f(D)}V = \frac{1}{f(D)}xV + \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V$$

or $\quad x\frac{1}{f(D)}V - \frac{f'(D)}{\{f(D)\}^2}V = \frac{1}{f(D)}xV$

ie., $\quad \left[x - \frac{f'(D)}{f(D)}\right]\frac{V}{f(D)} = \frac{1}{f(D)}xV$

Thus $\quad \frac{xV}{f(D)} = \left[x - \frac{f'(D)}{f(D)}\right]\frac{V}{f(D)}$

Note :

1. *By repeated application of this result we can find the particular integral of the functions* $x^2 V$, $x^3 V$ *etc.*

2. *In the cases of* $x^n \cos ax$ *and* $x^n \sin ax$ *specially when* $n > 1$ *we prefer the following* ***alternative method.***

We know that $e^{iax} = \cos ax + i \sin ax$

$\Rightarrow \quad \cos ax =$ Real part (R.P) of e^{iax}

$\sin ax =$ Imaginary part (I.P) of e^{iax}

Hence $x^n \cos ax = R.P(e^{iax}x^n)$ and $x^n \sin ax = I.P(e^{iax}x^n)$

Computation of P.I is done by first shifting D to $D + ia$ and then we have to divide x^n by the transformed form of $f(D)$. By separating this into real and imaginary parts we obtain the P.I of $x^n \cos ax/f(D)$ and $x^n \sin ax/f(D)$ respectively.

Illustrations

1. $\dfrac{x\cos x}{D^2+9}$

Here $V = \cos x$ and $f(D) = D^2+9 \therefore f'(D) = 2D$

$$\text{Now}\quad \frac{x\cos x}{D^2+9} = \left[x - \frac{2D}{D^2+9}\right]\frac{\cos x}{D^2+9}$$

$$= \left[x - \frac{2D}{D^2+9}\right]\frac{\cos x}{8}, \text{by replacing } D^2 \text{ by } -1$$

$$= \frac{x\cos x}{8} + \frac{\sin x}{4(D^2+9)} = \frac{x\cos x}{8} + \frac{\sin x}{4(-1+9)} = \frac{x\cos x}{8} + \frac{\sin x}{32}$$

$$\text{Thus}\quad \frac{x\cos x}{D^2+9} = \frac{1}{32}(4x\cos x + \sin x)$$

$$\textbf{Aliter}: \frac{x\cos x}{D^2+9} = R\cdot P\frac{x\,e^{ix}}{D^2+9} = R\cdot P\frac{e^{ix}\cdot x}{D^2+9}$$

$$ie., \quad = R.P\,e^{ix}\cdot\frac{x}{(D+i)^2+9} = R.P\,e^{ix}\frac{x}{D^2+2iD+8} \quad (\because i^2 = -1)$$

We divide x by $8+2iD+D^2$

$$\begin{array}{r|l} & x/8 - i/32 \\ \hline 8+2iD+D^2 & x \\ & x + i/4 \\ & (-)\ \ (-) \\ \hline & -i/4 \\ & -i/4 \\ & (+) \\ \hline & \quad 0 \end{array}$$

$$\text{Quotient} = \frac{x}{8} - \frac{i}{32} = \frac{1}{32}(4x - i)$$

$$P.I = R\cdot P\,e^{ix}\cdot\frac{1}{32}(4x-i)$$

$$= \frac{1}{32}R\cdot P\{(\cos x + i\sin x)(4x-i)\}$$

$$= \frac{1}{32}R\cdot P\{(4x\cos x + \sin x) + i(4x\sin x - \cos x)\}$$

$$P.I = \frac{1}{32}(4x\cos x + \sin x)$$

2.5 Summary of all the results

Solution of $f(D)\,y = \phi(x)$	
Stage-1	**Stage-2**
Finding C.F (y_c) by solving $f(D)\,y = 0$	Finding P.I (y_p) where $y_p = \phi(x)/f(D)$
Complete solution : $y = \text{C.F} + \text{P.I}$ or $y = y_c + y_p$	

Stage-1 : Solution of $f(D)\,y = 0$

- ➤ Form the A.E $f(m) = 0$ and solve.
- ➤ C.F is based on the nature of the roots of the A.E as summarized in the following table.

	Nature of the roots of the A.E	Complimentary Function (C.F)
1.	$m_1, m_2, \ldots m_n$ (real and distinct)	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$
2.	$m_1 = m_2 = \cdots = m_n = m$ (n coincident real roots)	$(c_1 + c_2 x + c_3 x^2 + \cdots + c_n x^{n-1})\, e^{mx}$
3.	(a) $p \pm iq$ (a pair of complex roots) (b) $\pm iq$ (a pair of imaginary roots) (c) $p \pm iq$ repeated n times	$e^{px}(c_1 \cos qx + c_2 \sin qx)$ $c_1 \cos qx + c_2 \sin qx$ $e^{px}\,[(c_1 + c_2 x + \cdots + c_n x^{n-1}) \cos qx$ $+ (c_1' + c_2' x + \cdots + c_n' x^{n-1}) \sin qx]$

Stage-2 : Particular solution of $f(D)y = \phi(x)$ or $P \cdot I = y_p = \phi(x)/f(D)$

	Type of P.I	**Method of getting the P.I**
1.	$\frac{e^{ax}}{f(D)}$ (Also applicable for e^{ax+b}, $a^x = e^{\log a \cdot x}$, sin*hax* cos*hax* by using their definitions)	$\frac{e^{ax}}{f(a)}$, $f(a) \neq 0$; (D is replaced by a) $x \cdot \frac{e^{ax}}{f'(a)}$, $x^2 \cdot \frac{e^{ax}}{f''(a)}$ etc. [if $f(a) = 0$], [if $f'(a) = 0$]
2.	$\frac{\sin ax}{f(D^2)}$, $\frac{\cos ax}{f(D^2)}$ (Also applicable for $\sin(ax+b)$, $\cos(ax+b)$ & $\sin ax \cos bx$, $\sin ax \sin bx$ $\cos ax \cos bx$ by changing into sum)	$\frac{\sin ax}{f(-a^2)}$, $\frac{\cos ax}{f(-a^2)}$ where $f(-a^2) \neq 0$ (D^2 is replaced by $-a^2$) $x \cdot \frac{\sin ax}{f'(-a^2)}$, $x \cdot \frac{\cos ax}{f'(-a^2)}$ if $f(-a^2) = 0$ etc.
3.	$\frac{\phi(x)}{f(D)}$ where $\phi(x)$ is a polynomial in x.	Divide $\phi(x)$ by $f(D)$ by writing $\phi(x)$ in descending powers of x and $f(D)$ in ascending powers of D. Quotient = P.I as remainder will be zero.
4.	$\frac{e^{ax}\,V}{f(D)}$ where $V = V(x)$	$e^{ax} \cdot \frac{V}{f(D+a)}$ D is replaced by $D+a$ first & then find $V/f(D+a)$ premultiplied by e^{ax}
5.	$\frac{xV}{f(D)}$; $V = V(x)$ $\frac{x^2 V}{f(D)}$, $\frac{x^n \cos ax}{f(D)}$, $\frac{x^n \sin ax}{f(D)}$	$\left[x - \frac{f'(D)}{f(D)}\right] \frac{V}{f(D)}$ $x\left[\frac{xV}{f(D)}\right]$ (Repeated application), $R.P.\ \frac{e^{iax}\,x^n}{f(D)}$, $I.P.\ \frac{e^{iax}\,x^n}{f(D)}$

WORKED PROBLEMS

Type-1

21. *Solve :* $6\frac{d^2y}{dx^2}+17\frac{dy}{dx}+12y = e^{-x}$

>> We have $(6D^2+17D+12)\,y = e^{-x}$

A.E is $6m^2+17m+12 = 0$

ie., $6m^2+9m+8m+12 = 0$

ie., $3m(2m+3)+4(2m+3) = 0$

ie., $(2m+3)(3m+4) = 0 \Rightarrow m = -3/2,\ -4/3$

$\therefore$ $y_c = c_1 e^{-3x/2}+c_2 e^{-4x/3}$

$$y_p = \frac{e^{-x}}{6D^2+17D+12} = \frac{e^{-x}}{6(-1)^2+17(-1)+12} = \frac{e^{-x}}{1} = e^{-x}$$

Complete solution : $y = y_c+y_p$

Thus $\boldsymbol{y = c_1 e^{-3x/2}+c_2 e^{-4x/3}+e^{-x}}$

- -

22. *Solve :* $y''+2y'+y = \cosh(x/2)$

>> We have $(D^2+2D+1)\,y = \cosh(x/2)$

A.E is $m^2+2m+1 = 0$ or $(m+1)^2 = 0 \Rightarrow m = -1,\ -1$

$\therefore$ $y_c = (c_1+c_2x)\,e^{-x}$

$$y_p = \frac{1}{2}\cdot\frac{e^{x/2}+e^{-x/2}}{(D+1)^2} = \frac{1}{2}\left[\frac{e^{x/2}}{(D+1)^2}+\frac{e^{-x/2}}{(D+1)^2}\right]$$

ie., $$= \frac{1}{2}\left[\frac{e^{x/2}}{(1/2+1)^2}+\frac{e^{-x/2}}{(-1/2+1)^2}\right]$$

$$y_p = \frac{1}{2}\cdot\frac{4}{9}e^{x/2}+\frac{1}{2}\cdot\frac{e^{-x/2}}{(1/4)} = \frac{2}{9}e^{x/2}+2e^{-x/2}$$

Complete solution : $y = y_c+y_p$

Thus $\boldsymbol{y = (c_1+c_2x)\,e^{-x}+2/9\cdot e^{x/2}+2e^{-x/2}}$

- -

23. *Solve :* $(D^3 - D^2 + 4D - 4)\, y = \sinh(2x + 3)$

>> A.E is $m^3 - m^2 + 4m - 4 = 0$

ie., $m^2(m-1) + 4(m-1) = 0$

ie., $(m-1)(m^2+4) = 0 \Rightarrow m = 1, \pm 2i$

$\therefore$ $y_c = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x$

$$y_p = \frac{\sinh(2x+3)}{D^3 - D^2 + 4D - 4} = \frac{1}{2}\left[\frac{e^{2x+3}}{D^3 - D^2 + 4D - 4} - \frac{e^{-(2x+3)}}{D^3 - D^2 + 4D - 4}\right]$$

ie., $$= \frac{1}{2}\left[\frac{e^{2x+3}}{8-4+8-4} - \frac{e^{-(2x+3)}}{-8-4-8-4}\right]$$

$$= \frac{1}{2}\left[\frac{e^{2x+3}}{8} - \frac{e^{-(2x+3)}}{-24}\right]$$

$$y_p = \frac{1}{48}\left[3\, e^{2x+3} + e^{-(2x+3)}\right]$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{48}\left[3\, e^{2x+3} + e^{-(2x+3)}\right]}$

- -

24. *Solve :* $y'' - (a+b)\, y' + aby = e^{ax} + e^{bx}$

>> We have $[D^2 - (a+b)D + ab]\, y = e^{ax} + e^{bx}$

A.E is $m^2 - (a+b)m + ab = 0$

or $(m-a)(m-b) = 0 \Rightarrow m = a$ and b

$\therefore$ $y_c = c_1 e^{ax} + c_2 e^{bx}$ where $a \neq b$

$$y_p = \frac{e^{ax} + e^{bx}}{(D-a)(D-b)} = \frac{e^{ax}}{(D-a)(D-b)} + \frac{e^{bx}}{(D-a)(D-b)} = p_1 + p_2$$

$$p_1 = \frac{e^{ax}}{(D-a)(D-b)} = \frac{e^{ax}}{(a-a)(a-b)} \quad (Dr. = 0)$$

$$= x \cdot \frac{e^{ax}}{(D-a)\cdot 1 + (D-b)\, 1} = x \cdot \frac{e^{ax}}{0 + (a-b)} = x \cdot \frac{e^{ax}}{a-b}$$

$$p_2 = \frac{e^{bx}}{(D-a)(D-b)} = \frac{e^{bx}}{(b-a)(b-b)} \quad (Dr. = 0)$$

$$= x \cdot \frac{e^{bx}}{(D-a)+(D-b)} = x \cdot \frac{e^{bx}}{(b-a)+0} = -x \cdot \frac{e^{bx}}{a-b}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2$

Thus $\boldsymbol{y = c_1 e^{ax} + c_2 e^{bx} + \frac{x}{a-b}(e^{ax} - e^{bx})}$ where $a \neq b$

25. *Solve* : $\frac{d^2x}{dt^2} - 6\frac{dx}{dt} + 9x = 5e^{-2t}$

\>> We have $(D^2 - 6D + 9)x = 5e^{-2t}$ where $D = \frac{d}{dt}$

A.E is $m^2 - 6m + 9 = 0$ or $(m-3)^2 = 0 \Rightarrow m = 3, 3$

$\therefore \quad x_c = (c_1 + c_2 t)e^{3t}$

$$x_p = \frac{5e^{-2t}}{D^2 - 6D + 9} = \frac{5e^{-2t}}{(-2)^2 - 6(-2) + 9} = \frac{5e^{-2t}}{25} = \frac{e^{-2t}}{5}$$

Complete solution : $x = x_c + x_p$

Thus $\boldsymbol{x = (c_1 + c_2 t)e^{3t} + e^{-2t}/5}$

26. *Solve* : $\frac{d^4x}{dt^4} + 4x = \cosh t$

\>> We have $(D^4 + 4)x = \cosh t$ where $D = \frac{d}{dt}$

A.E is $m^4 + 4 = 0$ or $(m^2+2)^2 - 4m^2 = 0$

or $\quad [(m^2+2) - 2m][(m^2+2) + 2m] = 0$

ie., $\quad m^2 - 2m + 2 = 0 \quad ; m^2 + 2m + 2 = 0$

$\therefore \quad m = \frac{2 \pm \sqrt{4-8}}{2} \quad ; m = \frac{-2 \pm \sqrt{4-8}}{2}$

$$m = \frac{2 \pm 2i}{2} = 1 \pm i \; ; \; m = \frac{-2 \pm 2i}{2} = -1 \pm i$$

$$\therefore \quad x_c = e^t(c_1\cos t + c_2\sin t) + e^{-t}(c_3\cos t + c_4\sin t)$$

$$x_p = \frac{\cosh t}{D^4+4} = \frac{1}{2}\left[\frac{e^t+e^{-t}}{D^4+4}\right] = \frac{1}{2}\left[\frac{e^t}{D^4+4}+\frac{e^{-t}}{D^4+4}\right]$$

ie., $$= \frac{1}{2}\left[\frac{e^t}{1+4}+\frac{e^{-t}}{1+4}\right] = \frac{1}{5}\cdot\frac{1}{2}(e^t+e^{-t}) = \frac{\cosh t}{5}$$

Complete solution : $x = x_c + x_p$

Thus $\boldsymbol{x = e^t(c_1\cos t + c_2\sin t) + e^{-t}(c_3\cos t + c_4\sin t) + \cosh t/5}$

27. *Solve :* $\dfrac{d^2y}{dx^2} - 4y = \cosh(2x-1) + 3^x$

>> We have $(D^2-4)y = \cosh(2x-1)+3^x$

A.E is $m^2-4 = 0$ or $(m-2)(m+2) = 0 \Rightarrow m = 2, -2$

$$\therefore \quad y_c = c_1e^{2x} + c_2e^{-2x}$$

$$y_p = \frac{\cosh(2x-1)+3^x}{D^2-4} = \frac{1}{2}\left[\frac{e^{2x-1}}{D^2-4}+\frac{e^{-(2x-1)}}{D^2-4}\right] + \frac{3^x}{D^2-4}$$

$$= p_1+p_2+p_3\ (say)$$

$$p_1 = \frac{1}{2}\cdot\frac{e^{2x-1}}{D^2-4} = \frac{1}{2}\frac{e^{2x-1}}{2^2-4} \quad (Dr. = 0)$$

$$= \frac{1}{2}x\frac{e^{2x-1}}{2D} = \frac{1}{2}\cdot x\frac{e^{2x-1}}{4} = \frac{x}{8}e^{2x-1}$$

$$p_2 = \frac{1}{2}\cdot\frac{e^{-(2x-1)}}{D^2-4} = \frac{1}{2}\cdot\frac{e^{-(2x-1)}}{(-2)^2-4}(Dr. = 0)$$

$$= \frac{1}{2}\cdot x\cdot\frac{e^{-(2x-1)}}{2D} = \frac{1}{2}\cdot x\cdot\frac{e^{-(2x-1)}}{-4} = \frac{-x}{8}e^{-(2x-1)}$$

$$\checkmark p_3 = \frac{3^x}{D^2-4} = \frac{(e^{\log 3})^x}{D^2-4} = \frac{e^{\log 3\cdot x}}{D^2-4} = \frac{e^{\log 3\cdot x}}{(\log 3)^2-4} = \frac{3^x}{(\log 3)^2-4}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1+p_2+p_3$

$$\therefore \qquad y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{8} e^{2x-1} - \frac{x}{8} e^{-(2x-1)} + \frac{3^x}{(\log 3)^2 - 4}$$

Thus $\boldsymbol{y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4} \sin h(2x-1) + \frac{3^x}{(\log 3)^2 - 4}}$

---

28. *Solve :* $\frac{d^2 y}{dt^2} - 4\frac{dy}{dt} + 13y = e^{3t} \cosh 2t + 2^t$

>> We have $(D^2 - 4D + 13)y = e^{3t} \cosh 2t + 2^t \;;\; D = \frac{d}{dt}$

A.E is $m^2 - 4m + 13 = 0$ and by solving,

$$m = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$\therefore \qquad y_c = e^{2t}(c_1 \cos 3t + c_2 \sin 3t)$$

$$y_p = \frac{e^{3t} \cosh 2t}{D^2 - 4D + 13} + \frac{2^t}{D^2 - 4D + 13}$$

$$= \frac{1}{2} \frac{e^{3t}(e^{2t} + e^{-2t})}{D^2 - 4D + 13} + \frac{2^t}{D^2 - 4D + 13}$$

$$= \frac{1}{2} \cdot \frac{e^{5t}}{D^2 - 4D + 13} + \frac{1}{2} \frac{e^t}{D^2 - 4D + 13} + \frac{2^t}{D^2 - 4D + 13}$$

$$= p_1 + p_2 + p_3 \text{ (say)}$$

$$p_1 = \frac{1}{2} \frac{e^{5t}}{D^2 - 4D + 13} = \frac{1}{2} \frac{e^{5t}}{(5)^2 - 4(5) + 13} = \frac{1}{2} \frac{e^{5t}}{18} = \frac{e^{5t}}{36}$$

$$p_2 = \frac{1}{2} \frac{e^t}{D^2 - 4D + 13} = \frac{1}{2} \frac{e^t}{1 - 4 + 13} = \frac{e^t}{20}$$

$$p_3 = \frac{2^t}{D^2 - 4D + 13} = \frac{e^{\log 2 \cdot t}}{D^2 - 4D + 13} = \frac{e^{\log 2 \cdot t}}{(\log 2)^2 - 4(\log 2) + 13}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2 + p_3$

Thus $\boldsymbol{y = e^{2t}(c_1 \cos 3t + c_2 \sin 3t) + \frac{e^{5t}}{36} + \frac{e^t}{20} + \frac{2^t}{(\log 2)^2 - 4(\log 2) + 13}}$

---

29. *Solve :* $(D^4 - 18D^2 + 81)\,y = 36\,e^{3x}$

>> A.E is $m^4 - 18m^2 + 81 = 0$

ie., $(m^2 - 9)^2 = 0$

or $(m-3)^2(m+3)^2 = 0$

$\Rightarrow$ $m = 3,\ 3\ ;\ -3,\ -3$

$\therefore$ $y_c = (c_1 + c_2 x)\,e^{3x} + (c_3 + c_4 x)\,e^{-3x}$

$$y_p = \frac{36\,e^{3x}}{D^4 - 18D^2 + 81} = \frac{36\,e^{3x}}{3^4 - 18(3^2) + 81} \quad (Dr. = 0)$$

$$= x \cdot \frac{36\,e^{3x}}{4D^3 - 36\,D} = x \cdot \frac{36\,e^{3x}}{4(3)^3 - 36(3)} \quad (Dr. = 0)$$

$$= x^2 \cdot \frac{36\,e^{3x}}{12\,D^2 - 36} = x^2 \cdot \frac{36\,e^{3x}}{12(3)^2 - 36} = x^2 \cdot \frac{36\,e^{3x}}{72} = \frac{x^2 e^{3x}}{2}$$

Complete solution : $y = y_c + y_p$

Thus $y = (c_1 + c_2 x)\,e^{3x} + (c_3 + c_4 x)\,e^{-3x} + x^2 e^{3x}/2$

---

30. *Solve :* $x'''(t) - 8x(t) = (1 - e^t)^2$

>> We have $(D^3 - 8)\,x(t) = (1 - e^t)^2$ where $D = \dfrac{d}{dt}$

A.E is $m^3 - 8 = 0$ or $(m-2)(m^2 + 2m + 4) = 0$

$m = 2\ ;\ m^2 + 2m + 4 = 0$ and by solving this equation

$$m = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm 2i\sqrt{3}}{2} = -1 \pm i\sqrt{3}$$

$\therefore$ $x_c = c_1 e^{2t} + e^{-t}(c_2 \cos\sqrt{3}\,t + c_3 \sin\sqrt{3}\,t)$

$$x_p = \frac{(1 - e^t)^2}{D^3 - 8} = \frac{1}{D^3 - 8} - \frac{2e^t}{D^3 - 8} + \frac{e^{2t}}{D^3 - 8} = p_1 + p_2 + p_3 \ (say)$$

$$p_1 = \frac{e^{0t}}{D^3 - 8} = \frac{e^{0t}}{-8} = \frac{-1}{8}$$

$$p_2 = \frac{-2e^t}{D^3 - 8} = \frac{-2e^t}{1 - 8} = \frac{2\,e^t}{7}$$

$$p_3 = \frac{e^{2t}}{D^3 - 8} = \frac{e^{2t}}{2^3 - 8} \quad (\,Dr. = 0\,)$$

$$= t \cdot \frac{e^{2t}}{3D^2} = t\,\frac{e^{2t}}{3 \cdot 2^2} = \frac{t\,e^{2t}}{12}$$

Complete solution : $x = x_c + x_p$ where $x_p = p_1 + p_2 + p_3$

Thus $\boldsymbol{x = c_1 e^{2t} + e^{-t}\,(\,c_2 \cos\sqrt{3}\,t + c_3 \sin\sqrt{3}\,t\,) - 1/8 + 2e^t/7 + te^{2t}/12}$

- -

Type-2

31. *Solve :* $y'' - 4y' + 13y = \cos 2x$

\>> We have $(\,D^2 - 4D + 13\,)\,y = \cos 2x$

A.E is $m^2 - 4m + 13 = 0$ and by solving,

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 13}}{2 \cdot 1} = \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm \sqrt{36i^2}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$\therefore \quad y_c = e^{2x}\,(\,c_1 \cos 3x + c_2 \sin 3x\,)$$

$$y_p = \frac{\cos 2x}{D^2 - 4D + 13}.$$

$-(a^2) = -4$

Here $a = 2$ and hence replace D^2 by $-a^2 = -4$

$$y_p = \frac{\cos 2x}{-4 - 4D + 13} = \frac{\cos 2x}{9 - 4D}.$$

Now multiply and divide by $(\,9 + 4D\,)$

$$= \frac{(\,9 + 4D\,)\cos 2x}{(\,9 + 4D\,)(\,9 - 4D\,)} = \frac{9\cos 2x + 4D\,(\,\cos 2x\,)}{81 - 16D^2}$$

$$= \frac{9\cos 2x - 8\sin 2x}{81 - 16\,(-4\,)} = \frac{9\cos 2x - 8\sin 2x}{145}$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = e^{2x}\,(\,c_1 \cos 3x + c_2 \sin 3x\,) + \frac{9\cos 2x - 8\sin 2x}{145}}$

- -

32. Solve : $y'' + 9y = \cos 2x \cdot \cos$

>> We have $(D^2+9)y = \cos 2x \cos x$

A.E is $m^2+9 = 0 \Rightarrow m = \pm 3i$

$\therefore \quad y_c = c_1 \cos 3x + c_2 \sin 3x$

$$y_p = \frac{1}{2}\frac{(\cos x + \cos 3x)}{D^2+9} = \frac{1}{2}\cdot\frac{\cos x}{D^2+9} + \frac{1}{2}\cdot\frac{\cos 3x}{D^2+9} = p_1 + p_2 \,(say)$$

$$p_1 = \frac{1}{2}\frac{\cos x}{D^2+9} = \frac{1}{2}\cdot\frac{\cos x}{-1^2+9} = \frac{\cos x}{16} \quad (D^2 \textit{ is replaced by } -a^2 = -1^2 = -1)$$

$$p_2 = \frac{1}{2}\frac{\cos 3x}{D^2+9} = \frac{1}{2}\cdot\frac{\cos 3x}{-3^2+9} \quad (Dr. = 0)$$

$$= \frac{1}{2}\cdot x\frac{\cos 3x}{2D} = \frac{x}{4}\int \cos 3x \, dx = \frac{x \sin 3x}{12}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2$

Thus $\quad \mathbf{y = c_1 \cos 3x + c_2 \sin 3x + \frac{\cos x}{16} + \frac{x \sin 3x}{12}}$

---

33. Solve : $(D^3 - 1)y = 3\cos$

>> A.E is $(m^3-1) = 0$ or $(m-1)(m^2+m+1) = 0$

$m = 1 \; ; \; m^2+m+1 = 0$ and by solving this equation,

$$m = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\therefore \quad y_c = c_1 e^x + e^{-x/2}\left\{c_2 \cos(\sqrt{3}\,x/2) + c_3 \sin(\sqrt{3}\,x/2)\right\}$$

$$y_p = \frac{3\cos 2x}{D^3-1} = \frac{3\cos 2x}{(D^2 \cdot D)-1} \quad \text{Now } D^2 \to -2^2 = -4$$

$$ie., \quad = \frac{3\cos 2x}{-4D-1} = -\frac{3(4D-1)\cos 2x}{-(4D+1)(4D-1)} = \frac{3(-8\sin 2x - \cos 2x)}{-(16D^2-1)}$$

$$y_p = \frac{-3(8\sin 2x + \cos 2x)}{65}$$

Complete solution : $y = y_c + y_p$

Thus $\quad y = c_1 e^x + e^{-x/2}\left\{c_2 \cos(\sqrt{3}\,x/2) + c_3 \sin(\sqrt{3}\,x/2)\right\} - \frac{3}{65}(8\sin 2x + \cos 2x)$

---

34. *Solve* : $\dfrac{d^2y}{dx^2}+3\dfrac{dy}{dx}+2y = 4\cos^2 x$

>> We have $(D^2+3D+2)\,y = 4\cos^2 x$

A.E is $m^2+3m+2 = 0$ or $(m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$\therefore \quad y_c = c_1 e^{-x}+c_2 e^{-2x}$

We have $4\cos^2 x = 2(1+\cos 2x)$

$$y_p = \frac{2}{D^2+3D+2}+\frac{2\cos 2x}{D^2+3D+2} = p_1+p_2 \text{ (say)}$$

$$p_1 = \frac{2e^{0x}}{D^2+3D+2} = \frac{2e^{0x}}{0^2+3\cdot 0+2} = 1$$

$$p_2 = \frac{2\cos 2x}{D^2+3D+2} = 2\left[\frac{\cos 2x}{-4+3D+2}\right] = 2\left[\frac{\cos 2x}{3D-2}\right]$$

$$= \frac{2(3D+2)(\cos 2x)}{(3D+2)(3D-2)} = \frac{2(-6\sin 2x+2\cos 2x)}{9D^2-4}$$

$$= \frac{4(-3\sin 2x+\cos 2x)}{9(-4)-4} = \frac{4(-3\sin 2x+\cos 2x)}{-40}$$

$$\therefore \quad p_2 = \frac{3\sin 2x-\cos 2x}{10}$$

Complete solution : $y = y_c+y_p$ where $y_p = p_1+p_2$

Thus $\boldsymbol{y = c_1 e^{-x}+c_2 e^{-2x}+1+\dfrac{3\sin 2x-\cos 2x}{10}}$

---

35. *Solve* : $(D^2+4)\,y = \sin^2 x$

>> A.E is $m^2+4 = 0 \Rightarrow m = \pm 2i$

$\therefore \quad y_c = c_1\cos 2x+c_2\sin 2x$

$$y_p = \frac{\sin^2 x}{D^2+4} = \frac{1}{2}\frac{(1-\cos 2x)}{D^2+4} = \frac{1}{2}\cdot\frac{1}{D^2+4}-\frac{1}{2}\frac{\cos 2x}{D^2+4} = p_1+p_2$$

$$p_1 = \frac{1}{2}\frac{e^{0x}}{D^2+4} = \frac{1}{2}\cdot\frac{e^{0x}}{0^2+4} = \frac{1}{8}$$

$$p_2 = -\frac{1}{2}\frac{\cos 2x}{D^2+4} \quad (D^2 \to -2^2)\,; \quad p_2 = \frac{-1}{2}\frac{\cos 2x}{-4+4} \quad (Dr. = 0)$$

$$\therefore \quad p_2 = -\frac{1}{2}x\frac{\cos 2x}{2D} = -\frac{1}{4}x\int \cos 2x\,dx = \frac{-x\sin 2x}{8}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2$

Thus $\boldsymbol{y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8} - \frac{x\sin 2x}{8}}$

36. *Solve* : $D^2(D^2+4)(D^2+9)y = 2\sin(x/2)\cos(x/2)$

\>> A.E is $m^2(m^2+4)(m^2+9) = 0$

Roots of the A.E are : $0,\ 0,\ \pm 2i,\ \pm 3i$

$$\therefore \quad y_c = (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x) + (c_5 \cos 3x + c_6 \sin 3x)$$

$$y_p = \frac{\sin x}{D^2(D^2+4)(D^2+9)} \quad \text{Now } D^2 \to -1^2 = -1$$

$$\therefore \quad y_p = \frac{\sin x}{(-1)(-1+4)(-1+9)} = \frac{\sin x}{-24}$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x) + (c_5 \cos 3x + c_6 \sin 3x) - \sin x/24}$

37. *Solve* : $(D^4 + 8D^2 + 16)y = 2\cos^2 x$

\>> A.E is $m^4 + 8m^2 + 16 = 0$

or $(m^2+4)^2 = 0 \Rightarrow m = \pm 2i,\ \pm 2i$ *(repeated imaginary roots)*

$$\therefore \quad y_c = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$$

Also $2\cos^2 x = 1 + \cos 2x$

$$y_p = \frac{1+\cos 2x}{D^4+8D^2+16} = \frac{1}{D^4+8D^2+16} + \frac{\cos 2x}{D^4+8D^2+16} = p_1 + p_2$$

$$p_1 = \frac{e^{0x}}{D^4+8D^2+16} = \frac{e^{0x}}{0+0+16} = \frac{1}{16}$$

$$p_2 = \frac{\cos 2x}{D^4+8D^2+16} \quad \text{Now } D^2 \to -2^2 = -4$$

$$p_2 = \frac{\cos 2x}{(-4)^2 + 8(-4) + 16} = \frac{\cos 2x}{32 - 32} \quad (Dr. = 0)$$

$$= x \cdot \frac{\cos 2x}{4D^3 + 16D} = \frac{x}{4} \cdot \frac{\cos 2x}{D^2 D + 4D} = \frac{x}{4} \cdot \frac{\cos 2x}{(-4D + 4D)} \quad (Dr. = 0)$$

$$= x^2 \cdot \frac{\cos 2x}{12D^2 + 16} = x^2 \cdot \frac{\cos 2x}{12(-4) + 16} = \frac{x^2 \cos 2x}{-32}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2$

Thus $\boldsymbol{y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x + \frac{1}{16} - \frac{x^2 \cos 2x}{32}}$

38. *Solve* : $\frac{d^2 x}{dt^2} + k^2 x = \cos(kt + c)$

>> We have $(D^2 + k^2)x = \cos(kt + c)$ where $D = \frac{d}{dt}$

A.E is $m^2 + k^2 = 0 \Rightarrow m = \pm ki$

$\therefore \quad x_c = c_1 \cos kt + c_2 \sin kt$

$x_p = \frac{\cos(kt + c)}{D^2 + k^2}$ Now $D^2 \to -k^2$ makes the denominator 0

Hence $x_p = t \cdot \frac{\cos(kt + c)}{2D} = \frac{t}{2}\int \cos(kt + c)\, dt$

ie., $\quad x_p = \frac{t}{2} \cdot \frac{\sin(kt + c)}{k} = \frac{t \sin(kt + c)}{2k}$

Complete solution : $x = x_c + x_p$

Thus $\boldsymbol{x = c_1 \cos kt + c_2 \sin kt + \frac{t \sin(kt + c)}{2k}}$

39. *Solve* : $\frac{d^3 y}{dx^3} + y = 65\cos(2x + 1)$

>> We have $(D^3 + 1)y = 65\cos(2x + 1)$

$$y_c = c_1 e^{-x} + e^{x/2}\left\{ c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x \right\} \quad \textit{(Refer Ex-5)}$$

$$y_p = \frac{65\cos(2x + 1)}{D^3 + 1} = \frac{65\cos(2x + 1)}{(D^2 \cdot D) + 1} \quad \text{Now } D^2 \to -4$$

$$y_p = \frac{65\cos(2x+1)}{-4D+1} = \frac{65(1+4D)\cos(2x+1)}{(1+4D)(1-4D)}$$

$$= \frac{65\{\cos(2x+1) - 8\sin(2x+1)\}}{1-16D^2}$$

$$= \frac{65\{\cos(2x+1) - 8\sin(2x+1)\}}{1-16(-4)}$$

$\therefore \quad y_p = \cos(2x+1) - 8\sin(2x+1).$

Complete solution : $y = y_c + y_p$

Thus $\quad y = c_1 e^{-x} + e^{x/2}\left\{c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x\right\}$

$$+ \cos(2x+1) - 8\sin(2x+1)$$

---

40. *Given the equation :* $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y + 37\sin 3x = 0$, *find the value of* y *when* $x = \pi/2$ *if it is given that* $y = 3$ *and* $\dfrac{dy}{dx} = 0$ *when* $x = 0$

>> We have $(D^2 + 2D + 10)y = -37\sin 3x$

A.E is $m^2 + 2m + 10 = 0$ and by solving,

$$m = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

$\therefore \quad y_c = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$

$$y_p = \frac{-37\sin 3x}{D^2 + 2D + 10} \quad \text{Now } D^2 \to -9$$

$$= \frac{-37\sin 3x}{2D+1} = \frac{-37(2D-1)\sin 3x}{(2D+1)(2D-1)}$$

$$= \frac{-37(6\cos 3x - \sin 3x)}{4D^2 - 1} = \frac{-37(6\cos 3x - \sin 3x)}{4(-9)-1}$$

$y_p = 6\cos 3x - \sin 3x$

Complete solution : $y = y_c + y_p$

Thus $\quad y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + 6\cos 3x - \sin 3x \quad \ldots (1)$

Now $$\frac{dy}{dx} = e^{-x}(-3c_1 \sin 3x + 3c_2 \cos 3x) - e^{-x}(c_1 \cos 3x + c_2 \sin 3x) - 18 \sin 3x - 3 \cos 3x \qquad \ldots (2)$$

Consider $y = 3$ when $x = 0$

Hence (1) becomes, $3 = c_1 + 6 \Rightarrow c_1 = -3$

Also $\frac{dy}{dx} = 0$ when $x = 0$

Hence (2) becomes, $0 = 3c_2 - c_1 - 3 \Rightarrow c_2 = 0$

Using these values in (1) we have,

$$y = -3e^{-x} \cos 3x + 6 \cos 3x - \sin 3x \qquad \ldots (3)$$

Now we shall find y when $x = \pi/2$. We have from (3)

$$y(\pi/2) = -3e^{-\pi/2} \cos(3\pi/2) + 6 \cos(3\pi/2) - \sin(3\pi/2)$$

But $\cos(3\pi/2) = 0$ and $\sin(3\pi/2) = -1$

$\therefore \quad y(\pi/2) = -(-1) = 1$

Thus **$y = 1$ when $x = \pi/2$**

Type-3

41. [illegible] $+ 2y = 12x^2$

>> We have $(D^2 + 3D + 2)y = 12x^2$

A.E is $m^2 + 3m + 2 = 0$ or $(m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$$\therefore \quad y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_p = \frac{12x^2}{D^2 + 3D + 2}$$

We need to divide for obtaining the P.I.

$$\begin{array}{r|l}
 & \underline{6x^2 - 18x + 21} \\
2 + 3D + D^2 & 12x^2 \\
 & \underline{12x^2 + 36x + 12} \\
 & \quad -36x - 12 \\
 & \quad \underline{-36x - 54} \\
 & \qquad\quad 42 \\
 & \qquad\quad \underline{42} \\
 & \qquad\quad 0
\end{array}$$

[**Note** : $3D(6x^2) = 3\frac{d}{dx}(6x^2) = 36x$

and $D^2(6x^2) = 12$]

Hence $y_p = 6x^2 - 18x + 21$

Complete solution : $y = y_c + y_p$

Thus $\mathbf{y = c_1 e^{-x} + c_2 e^{-2x} + 6x^2 - 18x + 21}$

---

42. *Solve :* $y'' + 2y' + y = 2x + x^2$

>> We have $(D^2 + 2D + 1)\, y = 2x + x^2$

A.E is $m^2 + 2m + 1 = 0$ or $(m+1)^2 = 0 \Rightarrow m = -1, -1$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^{-x}$

$$y_p = \frac{2x + x^2}{D^2 + 2D + 1} = \frac{x^2 + 2x}{1 + 2D + D^2}$$ P.I is found by division.

$$\begin{array}{r|l} & x^2 - 2x + 2 \\ \hline 1 + 2D + D^2 & x^2 + 2x \\ & x^2 + 4x + 2 \\ \hline & \quad -2x - 2 \\ & \quad -2x - 4 \\ \hline & \qquad 2 \\ & \qquad 2 \\ \hline & \qquad 0 \end{array}$$

$\therefore \quad y_p = x^2 - 2x + 2$

Complete solution : $y = y_c + y_p$

Thus $\mathbf{y = (c_1 + c_2 x)\, e^{-x} + (x^2 - 2x + 2)}$

---

43. *Solve :* $\mathbf{(D^3 + 8)\, y = x^4 + 2x + 1}$

>> A.E is $m^3 + 8 = 0$ or $(m+2)(m^2 - 2m + 4) = 0$

$$m = -2 \text{ and } m = \frac{2 \pm \sqrt{4 - 16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$$\therefore \quad y_c = c_1 e^{-2x} + e^x \left\{ c_2 \cos(\sqrt{3}\, x) + c_3 \sin(\sqrt{3}\, x) \right\}$$

$$y_p = \frac{x^4 + 2x + 1}{8 + D^3}$$

P.I is found by division.

$$\begin{array}{r|l}
 & x^4/8 - x/8 + 1/8 \\ \cline{2-2}
8 + D^3 & x^4 + 2x + 1 \\
 & x^4 + 3x \\ \cline{2-2}
 & \quad -x + 1 \\
 & \quad -x - 0 \\ \cline{2-2}
 & \qquad 1 \\
 & \qquad 1 \\ \cline{2-2}
 & \qquad 0
\end{array}
\qquad
\begin{array}{l}
D^3 (x^4/8) = 3x \\
\\
\therefore \quad y_p = \frac{1}{8} (x^4 - x + 1)
\end{array}$$

Complete solution : $y = y_c + y_p$

Thus $\quad y = c_1 e^{-2x} + e^x \left\{ c_2 \cos (\sqrt{3}\, x) + c_3 \sin (\sqrt{3}\, x) \right\} + \frac{1}{8} (x^4 - x + 1)$

41. *Solve :* $\dfrac{d^3 x}{dt^3} + 3 \dfrac{d^2 x}{dt^2} = 1 + t$

>> We have $(D^3 + 3D^2) x = 1 + t$ where $D = \dfrac{d}{dt}$

A.E is $m^3 + 3m^2 = 0$ or $m^2 (m + 3) = 0 \Rightarrow m = 0, 0, -3$

$\therefore \qquad x_c = c_1 + c_2 t + c_3 e^{-3t}$

$$x_p = \frac{1 + t}{D^3 + 3D^2} = \frac{t + 1}{3D^2 + D^3}$$

P.I is found by division.

$$\begin{array}{r|l}
 & t^3/18 + t^2/9 \\ \cline{2-2}
3D^2 + D^3 & t + 1 \\
 & t + 1/3 \\ \cline{2-2}
 & \quad 2/3 \\
 & \quad 2/3 \\ \cline{2-2}
 & \quad\; 0
\end{array}
\qquad
\begin{array}{l}
\dfrac{t}{3D^2} = \dfrac{1}{3} \displaystyle\int\!\!\int t \, dt \, dt = \dfrac{1}{3} \dfrac{t^3}{6} = \dfrac{t^3}{18} \\
D^3 (t^3/18) = 1/3 \\
\dfrac{2/3}{3D^2} = \dfrac{2}{9} \displaystyle\int\!\!\int d t \, dt = \dfrac{t^2}{9}
\end{array}$$

Hence $\quad x_p = \dfrac{t^3}{18} + \dfrac{t^2}{9} = \dfrac{t^2}{18} (t + 2)$

Complete solution : $x = x_c + x_p$

Thus $\quad x = c_1 + c_2 t + c_3 e^{-3t} + \dfrac{t^2 (t + 2)}{18}$

55. Solve: $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=x^2$

>> We have $(D^2+5D+6)\,y = x^2$

A.E is $m^2+5m+6 = 0$ or $(m+2)(m+3) = 0 \Rightarrow m = -2, -3$

$\therefore \quad y_c = c_1 e^{-2x}+c_2 e^{-3x}$

$$y_p = \frac{x^2}{D^2+5D+6} = \frac{x^2}{6+5D+D^2}$$

P.I is found by division.

$$\begin{array}{r|l l}
 & x^2/6-5x/18+19/108 & \\
\hline
6+5D+D^2 & x^2 & 5D\,(x^2/6) = 5x/3 \\
 & \underline{x^2+(5x/3)+(1/3)} & D^2\,(x^2/6) = 1/3 \\
 & -(5x/3)-(1/3) & \\
 & \underline{-(5x/3)-(25/18)} & 5D\,(-5x/18) = -25/18 \\
 & \quad 19/18 & \\
 & \quad \underline{19/18} & \\
 & \quad 0 &
\end{array}$$

$$\therefore \quad y_p = \frac{x^2}{6}-\frac{5x}{18}+\frac{19}{108} = \frac{1}{108}(18x^2-30x+19)$$

Complete soluton : $y = y_c+y_p$

Thus $\boldsymbol{y = c_1 e^{-2x}+c_2 e^{-3x}+\frac{1}{108}(18x^2-30x+19)}$

- -

56. Solve $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}+\frac{dy}{dx}=x^3$

>> We have $(D^3+2D^2+D)\,y = x^3$

A.E is $m^3+2m^2+m = 0$ or $m(m^2+2m+1) = 0$

ie., $\quad m(m+1)^2 = 0 \Rightarrow m = 0, -1, -1$

$\therefore \quad y_c = c_1+(c_2+c_3x)\,e^{-x}$

$$y_p = \frac{x^3}{D^3+2D^2+D} = \frac{x^3}{D+2D^2+D^3}$$

P.I is found by division.

$$\begin{array}{r|l|l}
 & x^4/4 - 2x^3 + 9x^2 - 24x & \\ \hline
D+2D^2+D^3 & x^3 & \frac{x^3}{D} = \int x^3\,dx = \frac{x^4}{4} \\
 & x^3+6x^2+6x & 2D^2(x^4/4) = D^2(x^4/2) = 6x^2 \\ \cline{2-2}
 & -6x^2-6x & D^3(x^4/4) = 6x \\
 & -6x^2-24x-12 & \frac{-6x^2}{D} = \int -6x^2\,dx = -2x^3 \\ \cline{2-2}
 & 18x+12 & \frac{18x}{D} = \int 18\,x\,dx = 9x^2 \\
 & 18x+36 & \\ \cline{2-2}
 & -24 & \frac{-24}{D} = \int -24\,dx = -24x \\
 & -24 & \\ \cline{2-2}
 & 0 &
\end{array}$$

$$\therefore \quad y_p = \frac{x^4}{4} - 2x^3 + 9x^2 - 24x$$

Complete solution : $y = y_c + y_p$

Thus $\mathbf{y = c_1 + (c_2 + c_3\,x)\,e^{-x} + (x^4/4) - 2x^3 + 9x^2 - 24x}$

---

47. *Solve :* $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} + 6y = 1 - x + x^2$

>> We have $(D^3 - 7D + 6)\,y = 1 - x + x^2$

A.E is given by $m^3 - 7m + 6 = 0$

$m = 1$ is a root by inspection.

We now apply the method of synthetic division

$$\begin{array}{r|rrrr}
1 & 1 & 0 & -7 & 6 \\
 & 0 & 1 & 1 & -6 \\ \hline
 & 1 & 1 & -6 & \boxed{0}
\end{array}$$

$m^2 + m - 6 = 0$ or $(m+3)(m-2) = 0 \Rightarrow m = -3, 2$

$m = 1, 2, -3$ are the roots of A.E

$$\therefore \quad y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-3x}$$

$$y_p = \frac{1 - x + x^2}{D^3 - 7D + 6} = \frac{x^2 - x + 1}{6 - 7D + D^3}$$

P.I is found by division.

$$\begin{array}{r|l}
 & x^2/6 + 2x/9 + 23/54 \\
\hline
6 - 7D + D^3 & x^2 - x + 1 \\
 & x^2 - \frac{7x}{3} + 0 \\
\hline
 & \frac{4x}{3} + 1 \\
 & \frac{4x}{3} - \frac{14}{9} \\
\hline
 & \frac{23}{9} \\
 & \frac{23}{9} \\
\hline
 & 0
\end{array}$$

$$y_p = \frac{x^2}{6} + \frac{2x}{9} + \frac{23}{54} = \frac{1}{54}\,(\,9x^2 + 12x + 23\,)$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-3x} + \frac{1}{54}\,(\,9x^2 + 12x + 23\,)}$

- -

48. Solve : $\frac{d^3 y}{dx^3} + \frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 4y = x^2 - 4x - 6$

>> We have $(\,D^3 + D^2 + 4D + 4\,)\; y = x^2 - 4x - 6$

A.E is given by $m^3 + m^2 + 4m + 4 = 0$

or $\quad m^2\,(\,m + 1\,) + 4\,(\,m + 1\,) = 0$

or $\quad (\,m + 1\,)\,(\,m^2 + 4\,) = 0 \;\Rightarrow\; m = -1\,,\; m = \pm\, 2i$

$$y_c = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x$$

$$y_p = \frac{x^2 - 4x - 6}{D^3 + D^2 + 4D + 4} \quad \text{P.I is found by division.}$$

$$\begin{array}{r|l}
 & x^2/4 - 3x/2 - 1/8 \\
\hline
4 + 4D + D^2 + D^3 & x^2 - 4x - 6 \\
 & x^2 + 2x + 1/2 \\
\hline
 & \quad -6x - (13/2) \\
 & \quad -6x - 6 \\
\hline
 & \qquad -1/2 \\
 & \qquad -1/2 \\
\hline
 & \qquad 0
\end{array}$$

Hence $y_p = \frac{x^2}{4} - \frac{3x}{2} - \frac{1}{8} = \frac{1}{8}(2x^2 - 12x - 1)$

Complete solution : $y = y_c + y_p$

Thus $\mathbf{y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{8}(2x^2 - 12x - 1)}$

Type-4

49. *Solve* : $(D^2 - 2D + 5)y = e^{2x} \sin x$

>> A.E is $m^2 - 2m + 5 = 0$ and $m = 1 \pm 2i$ by solving.

$\therefore \quad y_c = e^x (c_1 \cos 2x + c_2 \sin 2x)$

Now $y_p = \frac{e^{2x} \sin x}{D^2 - 2D + 5}$ First we replace D by $D + 2$

$$y_p = e^{2x}\left[\frac{\sin x}{(D+2)^2 - 2(D+2) + 5}\right] = e^{2x}\left[\frac{\sin x}{D^2 + 2D + 5}\right]$$

Now $D^2 \to -1^2 = -1$

$$y_p = e^{2x}\left[\frac{\sin x}{2D + 4}\right] = \frac{e^{2x}}{2}\left[\frac{\sin x}{D + 2}\right] = \frac{e^{2x}}{2}\,\frac{(D-2)\sin x}{(D-2)(D+2)}$$

$$y_p = \frac{e^{2x}}{2}\,\frac{\cos x - 2\sin x}{D^2-4} = \frac{e^{2x}}{2}\cdot\frac{\cos x - 2\sin x}{-1-4}$$

$$\therefore \quad y_p = \frac{e^{2x}}{10}(2\sin x - \cos x)$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = e^x(c_1\cos 2x + c_2\sin 2x) + \frac{e^{2x}}{10}(2\sin x - \cos x)}$

---

50. Solve $(D^2 - 2D + 4)\,y = e^x\cos x$

>> A.E is $m^2 - 2m + 4 = 0$ and $m = 1 \pm i\sqrt{3}$ by solving .

$$\therefore \quad y_c = e^x(c_1\cos\sqrt{3}\,x + c_2\sin\sqrt{3}\,x)$$

$$y_p = \frac{e^x\cos x}{D^2-2D+4} \quad \text{Now } D \to D+1$$

$$y_p = e^x\left[\frac{\cos x}{D^2+3}\right] \quad \text{Now } D^2 \to -1^2 = -1$$

$$\therefore \quad y_p = \frac{e^x\cos x}{2}$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = e^x(c_1\cos\sqrt{3}\,x + c_2\sin\sqrt{3}\,x) + e^x\cos x/2}$

---

51. Solve $(D^3 + 1)\,y = 5e^x x^2$

>> $y_c = c_1 e^{-x} + e^{x/2}\left\{c_2\cos(\sqrt{3}/2)\,x + c_3\sin(\sqrt{3}/2)\,x\right\}$ *(Refer Problem-5)*

$$y_p = e^x\,\frac{5x^2}{D^3+1} \quad \text{First } D \to D+1$$

$$= e^x\,\frac{5x^2}{(D+1)^3+1} = e^x\,\frac{5x^2}{D^3+3D^2+3D+2}$$

$$= \frac{5e^x}{2}\left[\frac{2x^2}{2+3D+3D^2+D^3}\right]$$

(For a convenient division we have multiplied and divided by 2)

$$\begin{array}{l|l} & x^2 - 3x + (3/2) \\ \hline 2 + 3D + 3D^2 + D^3 & 2x^2 \\ & 2x^2 + 6x + 6 \\ \hline & \quad -6x - 6 \\ & \quad -6x - 9 \\ \hline & \qquad 3 \\ & \qquad 3 \\ \hline & \qquad 0 \end{array}$$

$$y_p = \frac{5e^x}{2} \cdot [\, x^2 - 3x + (3/2) \,] = \frac{5e^x}{4} (\, 2x^2 - 6x + 3 \,)$$

Complete solution : $y = y_c + y_p$

Thus $y = c_1 e^{-x} + e^{x/2} \left\{ c_2 \cos (\sqrt{3}/2)\, x + c_3 \sin (\sqrt{3}/2)\, x \right\} + \frac{5e^x}{4} (\, 2x^2 - 6x + 3 \,)$

- -

52. *Solve :* $(D^2 - 4D + 3)\, y = 2\, x\, e^{3x}$

A.E is $m^2 - 4m + 3 = 0$ or $(m-1)(m-3) = 0 \Rightarrow m = 1, 3$

$\therefore \quad y_c = c_1 e^x + c_2 e^{3x}$

$$y_p = \frac{2\, e^{3x}\, x}{(D-1)(D-3)} \qquad \text{Now } D \to D + 3$$

$$y_p = 2\, e^{3x} \cdot \frac{x}{(D+2)\, D} = e^{3x} \cdot \frac{2x}{2D + D^2}$$

P.I. is found by division.

$$\begin{array}{l|l} & (x^2/2) - (x/2) \\ \hline 2D + D^2 & 2x \\ & 2x + 1 \\ \hline & \quad -1 \\ & \quad -1 \\ \hline & \quad 0 \end{array}$$

Note : $\frac{2x}{2D} = \int x\, dx = \frac{x^2}{2}$

$$\frac{-1}{2D} = \int -\frac{1}{2} \cdot 1\, dx = -\frac{x}{2}$$

$$\therefore \; y_p = e^{3x} \cdot \left(\frac{x^2}{2} - \frac{x}{2} \right)$$

Complete solution : $y = y_c + y_p$

Thus $y = c_1 e^x + c_2 e^{3x} + e^{3x} (\, x^2 - x \,)/2$

- -

53. *Solve* : $y'' + 2y' + 5y = e^{-x}\sin 2x$

>> We have $(D^2 + 2D + 5)\,y = e^{-x}\sin 2x$

A.E is $m^2 + 2m + 5 = 0$ and $m = -1 \pm 2i$, by solving.

$$\therefore \quad y_c = e^{-x}(c_1 \cos 2x + c_2 \sin 2x)$$

$$y_p = \frac{e^{-x}\sin 2x}{D^2 + 2D + 5} \text{ or } \frac{e^{-x}\sin 2x}{(D+1)^2 + 4} \quad \text{First } D \to D - 1$$

$$y_p = e^{-x} \cdot \frac{\sin 2x}{D^2 + 4} \quad \text{Now } D^2 \to -2^2 = -4$$

$$= e^{-x} \cdot \frac{\sin 2x}{-4 + 4} \,(Dr. = 0)$$

$$= e^{-x}\,x \cdot \frac{\sin 2x}{2D} = \frac{e^{-x} \cdot x}{2}\int \sin 2x\,dx = \frac{-e^{-x} \cdot x\cos 2x}{4}$$

Complete soultion : $y = y_c + y_p$

Thus $\boldsymbol{y = e^{-x}(c_1 \cos 2x + c_2 \sin 2x) - \dfrac{e^{-x}\,x\cos 2x}{4}}$

- -

54. *Solve* : $x''(t) - 4x(t) = t\sinh t$

>> We have $(D^2 - 4)\,x(t) = t\sinh t$, where $D = \dfrac{d}{dt}$

A.E is $m^2 - 4 = 0 \Rightarrow m = \pm 2$

$$\therefore \quad x_c = c_1 e^{2t} + c_2 e^{-2t}$$

$$x_p = \frac{t\sinh t}{D^2 - 4} = \frac{1}{2}\frac{t \cdot (e^t - e^{-t})}{D^2 - 4}$$

$$x_p = \frac{1}{2}\left[\frac{e^t \cdot t}{D^2 - 4} - \frac{e^{-t}t}{D^2 - 4}\right] = \frac{1}{2}[p_1 - p_2]\,(say)$$

$$p_1 = \frac{e^t \cdot t}{D^2 - 4} = e^t \cdot \frac{t}{(D+1)^2 - 4} = e^t \cdot \frac{t}{D^2 + 2D - 3}$$

$$\begin{array}{r|l} & (t/-3)-(2/9) \\ \hline -3+2D+D^2 & t \\ & t-(2/3) \\ \hline & \quad 2/3 \\ & \quad 2/3 \\ \hline & \quad 0 \end{array} \qquad \therefore\ p_1=\left(\frac{-t}{3}-\frac{2}{9}\right)e^t$$

$$p_2=\frac{e^{-t}\,t}{D^2-4}=e^{-t}\cdot\frac{t}{(D-1)^2-4}=e^{-t}\cdot\frac{t}{D^2-2D-3}$$

$$\begin{array}{r|l} & (t/-3)+(2/9) \\ \hline -3-2D+D^2 & t \\ & t+(2/3) \\ \hline & \quad -2/3 \\ & \quad -2/3 \\ \hline & \quad 0 \end{array} \qquad \therefore\ p_2=\left(\frac{-t}{3}+\frac{2}{9}\right)e^{-t}$$

$$x_p=\frac{1}{2}[p_1-p_2]$$

$$=\frac{1}{2}\left[-\left(\frac{t}{3}+\frac{2}{9}\right)e^t-\left(\frac{-t}{3}+\frac{2}{9}\right)e^{-t}\right]$$

$$=\frac{-t}{3}\left(\frac{e^t-e^{-t}}{2}\right)-\frac{2}{9}\left(\frac{e^t+e^{-t}}{2}\right)$$

ie., $x_p=\frac{-t}{3}\sinh t-\frac{2}{9}\cosh t$

Complete solution : $x=x_c+x_p$

Thus $x=c_1e^{2t}+c_2e^{-2t}-\frac{t\sinh t}{3}-\frac{2\cosh t}{9}$

---

55. [illegible]

>> We have $(D^4-1)y=e^t\cos t$, where $D=\frac{d}{dt}$

A.E is $m^4-1=0$ or $(m^2-1)(m^2+1)=0$

ie., $(m-1)(m+1)(m^2+1)=0 \Rightarrow m=1,\ -1,\ \pm i$

$\therefore$ $y_c=c_1e^t+c_2e^{-t}+c_3\cos t+c_4\sin t.$

$$y_p = \frac{e^t \cos t}{D^4 - 1} = e^t \cdot \frac{\cos t}{(D+1)^4 - 1}$$

$$= e^t \cdot \frac{\cos t}{D^4 + 4D^3 + 6D^2 + 4D + 1 - 1}$$

$$= e^t \cdot \frac{\cos t}{D^4 + 4D^3 + 6D^2 + 4D} \quad \text{Now } D^2 \to -1$$

$$y_p = e^t \cdot \frac{\cos t}{(-1)^2 + 4(-1)D + 6(-1) + 4D} = e^t \cdot \frac{\cos t}{-5}$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t - e^t \cos t/5}$

Type-5

56. *Solve :* $y'' + 16y = x \sin 3x$

>> We have $(D^2 + 16)y = x \sin 3x$

A.E is $m^2 + 16 = 0$ or $m = \sqrt{-16} = \sqrt{16i^2} = \pm 4i$

$\therefore \quad y_c = c_1 \cos 4x + c_2 \sin 4x$

$$y_p = \frac{x \sin 3x}{D^2 + 16}$$

Let us use $\dfrac{xV}{f(D)} = \left[x - \dfrac{f'(D)}{f(D)}\right]\dfrac{V}{f(D)}$

$$= \left[x - \frac{2D}{D^2+16}\right]\frac{\sin 3x}{D^2+16} = \left[x - \frac{2D}{D^2+16}\right]\frac{\sin 3x}{-3^2+16}$$

ie.,

$$= \frac{x \sin 3x}{7} - \frac{6 \cos 3x}{7(D^2+16)} = \frac{x \sin 3x}{7} - \frac{6 \cos 3x}{7(-3^2+16)}$$

$$y_p = \frac{x \sin 3x}{7} - \frac{6 \cos 3x}{49} = \frac{1}{49}[7x \sin 3x - 6 \cos 3x]$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{49}(7x \sin 3x - 6 \cos 3x)}$

---

57. *Solve :* $y'' - 2y' + y = x\cos x$

>> We have $(D^2 - 2D + 1)\,y = x\cos x$

A.E is $m^2 - 2m + 1 = 0$ or $(m-1)^2 = 0 \Rightarrow m = 1,\ 1$

$$\therefore \quad y_c = (c_1 + c_2 x)\,e^x$$

$$y_p = \frac{x\cos x}{D^2 - 2D + 1}$$

We have $\dfrac{xV}{f(D)} = \left[x - \dfrac{f'(D)}{f(D)}\right]\dfrac{V}{f(D)}$

Hence $\quad y_p = \left[x - \dfrac{(2D-2)}{D^2 - 2D + 1}\right]\dfrac{\cos x}{D^2 - 2D + 1}$

$$= \left[x - \frac{2(D-1)}{(D-1)^2}\right]\frac{\cos x}{-1 - 2D + 1}$$

$$= \left[x - \frac{2}{D-1}\right]\left(\frac{\sin x}{-2}\right) \because \quad \frac{\cos x}{-2D} = \int\frac{\cos x}{-2}\,dx = \frac{\sin x}{-2}$$

$$= \frac{-x\sin x}{2} + \frac{\sin x}{D-1}$$

$$= \frac{-x\sin x}{2} + \frac{(D+1)\sin x}{D^2 - 1}$$

$$= \frac{-x\sin x}{2} + \frac{\cos x + \sin x}{-2} = \frac{-1}{2}\,(x\sin x + \cos x + \sin x)$$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = (c_1 + c_2 x)\,e^x - \frac{1}{2}(x\sin x + \cos x + \sin x)}$

---

58. *Solve : (a)* $y'' - y = x^2\cos x$ *(b)* $y'' - y = x^2\sin x$

>> (a) We have $(D^2 - 1)\,y = x^2\cos x$

A.E is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$\therefore \quad y_c = c_1 e^x + c_2 e^{-x}$$

$$y_p = \frac{x^2\cos x}{D^2 - 1} = \frac{R.P\,(x^2 e^{ix})}{D^2 - 1}$$

[**Note** : *We do not prefer to apply the result of* $\frac{x\,V}{f(D)}$ *twice*]

$$y_p = R\,.\,P\ e^{ix}\frac{x^2}{D^2-1} \quad \text{We replace } D \text{ by } D+i$$

$$= R\,.\,P\ e^{ix}\frac{x^2}{(D+i)^2-1}$$

$$= R\,.\,P\,e^{ix}\frac{x^2}{D^2+2iD+i^2-1}$$

$$= R\,.\,P\,e^{ix}\frac{x^2}{D^2+2iD-2} = R\,.\,P\,e^{ix}\frac{x^2}{-2+2iD+D^2}$$

We need to employ division now.

$$\begin{array}{r|l}
 & (x^2/-2) - ix + (1/2) \\ \hline
-2+2i\,D+D^2 & x^2 \\
 & x^2 - 2i\,x - 1 \\ \hline
 & \quad 2i\,x+1 \\
 & \quad 2i\,x+2 \\ \hline
 & \qquad -1 \\
 & \qquad -1 \\ \hline
 & \qquad 0
\end{array}$$

$2i\,D\,(x^2/-2) = -2i\,x$

$D^2\,(x^2/-2) = -1$

Quotient is

$\frac{-1}{2}(x^2-1) - ix$

$$\therefore \quad y_p = R\,.\,P\,e^{ix}\left[\frac{1}{2}(1-x^2) - ix\right]$$

$$= R\,.\,P\,(\cos x + i\sin x)\left[\frac{1}{2}(1-x^2) - ix\right]$$

$$= R\,.\,P\left\{\left[\frac{1}{2}\cos x\,(1-x^2) + x\sin x\right] + i\left[\frac{1}{2}\sin x\,(1-x^2) - x\cos x\right]\right\}$$

Required $y_p = \frac{1}{2}\cos x\,(1-x^2) + x\sin x$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = c_1\,e^{x} + c_2\,e^{-x} + \frac{\cos x}{2}(1-x^2) + x\sin x}$

(b) $(D^2-1)\,y = x^2\sin x$

The working is same as in case (a) where

$$y_p = I \cdot P \ \frac{x^2 e^{ix}}{D^2 - 1}$$

Required $y_p = \frac{\sin x}{2}(1 - x^2) - x \cos x$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = c_1 e^x + c_2 e^{-x} + \frac{\sin x}{2}(1 - x^2) - x \cos x}$

---

59. *Solve :* $y'' - 2y' + y = x e^x \sin x$

>> We have $(D^2 - 2D + 1)\, y = x e^x \sin x$

A.E. is $m^2 - 2m + 1 = 0$ or $(m - 1)^2 = 0 \Rightarrow m = 1, 1$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^x$

$$y_p = e^x \frac{x \sin x}{(D-1)^2} \quad \text{First } D \to D+1$$

$$y_p = e^x \left[\frac{x \sin x}{D^2}\right] \quad \text{and we have } \frac{xV}{f(D)} = \left[x - \frac{f'(D)}{f(D)}\right] \frac{V}{f(D)}$$

Hence $y_p = e^x \left[x - \frac{2D}{D^2}\right] \frac{\sin x}{D^2} = e^x \left[x - \frac{2}{D}\right] \frac{\sin x}{-1}$

$$= e^x(-x \sin x + 2 \cdot - \cos x)$$

$\therefore \quad y_p = -e^x(x \sin x + 2 \cos x)$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = (c_1 + c_2 x)\, e^x - e^x(x \sin x + 2 \cos x)}$

---

60. (a) *Solve :* $y'' - 4y' + 4y = 8x^2 e^{2x} \cos 2x$

(b) *Solve :* $y'' - 4y' + 4y = 8x^2 e^{2x} \sin 2x$

>> (a) We have $(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \cos 2x$

A.E is $m^2 - 4m + 4 = 0$ or $(m - 2)^2 = 0 \Rightarrow m = 2, 2$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^{2x}$

$$y_p = \frac{8x^2 e^{2x} \cos 2x}{(D-2)^2} = e^{2x} \cdot \frac{8x^2 \cos 2x}{(D-2)^2} \; ; \; D \to D+2$$

ie., $$= e^{2x} \cdot \frac{8x^2 \cos 2x}{D^2}$$

Note : *P.I can also be completed by integrating* $8x^2 \cos 2x$ *twice by parts*

$$y_p = (e^{2x})\, R\,.\,P \frac{8x^2 e^{2ix}}{D^2} = (e^{2x})\, R\,.\,P \;\; e^{2ix} \; \frac{8x^2}{D^2}$$

$$= e^{2x} R\,.\,P\, e^{2ix} \frac{8x^2}{(D+2i)^2}$$

$$= e^{2x} R\,.\,P\, e^{2ix} \frac{8x^2}{D^2+4iD-4}$$

We need to employ division now.

$$\begin{array}{r|l} & -2x^2-4i\,x+3 \\ \cline{2-2} -4+4\,i\,D+D^2 & 8x^2 \\ & 8x^2-16i\,x-4 \\ \cline{2-2} & \quad 16i\,x+4 \\ & \quad 16i\,x+16 \\ \cline{2-2} & \qquad -12 \\ & \qquad -12 \\ \cline{2-2} & \qquad\; 0 \end{array}$$

Quotient is $-2x^2-4ix+3$

$$y_p = e^{2x} R\,.\,P\, e^{2ix} (-2x^2-4ix+3)$$

$$= e^{2x} R\,.\,P\,(\cos 2x + i \sin 2x)(-2x^2+3-4ix)$$

$$= e^{2x} R\,.\,P\, \{ [\cos 2x(-2x^2+3) + 4x \sin 2x]$$
$$+ i [\sin 2x(-2x^2+3) - 4x \cos 2x] \}$$

Thus $y_p = e^{2x} [\cos 2x (3-2x^2) + 4x \sin 2x]$

Complete solution : $y = y_c + y_p$

Thus $\boldsymbol{y = (c_1 + c_2 x) e^{2x} + e^{2x} [\cos 2x (3-2x^2) + 4x \sin 2x]}$

(b) P.I in respect of R.H.S being $8x^2 e^{2x} \sin 2x$ will be
$e^{2x} [\sin 2x (3 - 2x^2) - 4x \cos 2x]$ by taking the imaginary part.

Thus $\boldsymbol{y = (c_1 + c_2 x) e^{2x} + e^{2x} [\sin 2x (3 - 2x^2) - 4x \cos 2x]}$

Type 6 .. Mixed Type of Problems

61. *Solve :* $(D^3 + 1) y = \cos(\pi/2 - x) + e^x$

\>> $y_c = c_1 e^{-x} + e^{x/2} \{c_2 \cos(\sqrt{3}\, x/2) + c_3 \sin(\sqrt{3}\, x/2)\}$ *(Refer Problem-5)*

$$y_p = \frac{\sin x + e^x}{D^3 + 1} = \frac{\sin x}{D^3 + 1} + \frac{e^x}{D^3 + 1} = p_1 + p_2$$

$$p_1 = \frac{\sin x}{D^3 + 1} = \frac{\sin x}{D^2 \cdot D + 1} \quad \text{Now } D^2 \to -1^2 = -1$$

$$p_1 = \frac{\sin x}{1 - D} = \frac{(1 + D) \sin x}{(1 + D)(1 - D)} = \frac{\sin x + \cos x}{1 - D^2} = \frac{\sin x + \cos x}{2}$$

$$p_2 = \frac{e^x}{D^3 + 1} \quad \text{Here } D \to 1 \quad \therefore \quad p_2 = \frac{e^x}{2}$$

Complete solution : $y = y_c + y_p$ where

Thus $\boldsymbol{y = c_1 e^{-x} + e^{x/2} \{c_2 \cos(\sqrt{3}\, x/2) + c_3 \sin(\sqrt{3}\, x/2)\} + \frac{1}{2} (\sin x + \cos x + e^x)}$

---

62. *Solve :* $\dfrac{d^2 y}{dx^2} - 2 \dfrac{dy}{dx} + y = x e^x + x$

\>> We have $(D^2 - 2D + 1) y = x e^x + x$

A.E is $m^2 - 2m + 1 = 0$ or $(m - 1)^2 = 0 \Rightarrow m = 1, 1$

$$\therefore \quad y_c = (c_1 + c_2 x) e^x$$

$$y_p = \frac{x e^x}{D^2 - 2D + 1} + \frac{x}{D^2 - 2D + 1} = p_1 + p_2 \,(say)$$

$$p_1 = e^x \cdot \frac{x}{(D - 1)^2} \; ; \; D \to D + 1$$

$$\therefore \quad p_1 = e^x \cdot \frac{x}{D^2} = e^x \iint x \, dx \, dx = e^x \cdot \frac{x^3}{6} = \frac{x^3 e^x}{6}$$

$$p_2 = \frac{x}{1-2D+D^2} \text{ and we shall divide.}$$

$$\begin{array}{r|l} & x+2 \\ \hline 1-2D+D^2 & x \\ & x-2 \\ \hline & 2 \\ & 2 \\ \hline & 0 \end{array} \qquad \therefore \; p_2 = x+2$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2$

Thus $y = (c_1 + c_2 x) e^x + x^3 e^x/6 + (x+2)$

---

63. *Solve :* $y'' + 4y' \;\; 12y = e^{2x} \;\; 3\sin 2x$

>> We have $(D^2 + 4D - 12)\, y = e^{2x} - 3\sin 2x$

A.E is $m^2 + 4m - 12 = 0$ or $(m+6)(m-2) = 0 \Rightarrow m = 2, -6$

$$\therefore \quad y_c = c_1 e^{2x} + c_2 e^{-6x}$$

$$y_p = \frac{e^{2x}}{D^2+4D-12} - \frac{3\sin 2x}{D^2+4D-12} = p_1 - p_2\,(say)$$

$$p_1 = \frac{e^{2x}}{D^2+4D-12} = \frac{e^{2x}}{4+8-12} \qquad (Dr. = 0)$$

$$= x \cdot \frac{e^{2x}}{2D+4} = x \cdot \frac{e^{2x}}{8} = \frac{x\,e^{2x}}{8}$$

$$p_2 = \frac{3\sin 2x}{D^2+4D-12} \; ; \; D^2 \to -4$$

$$p_2 = \frac{3\sin 2x}{-4+4D-12} = \frac{3\sin 2x}{4(D-4)} = \frac{3(D+4)\sin 2x}{4(D^2-16)}$$

$$\therefore \quad p_2 = \frac{3(2\cos 2x + 4\sin 2x)}{-80}$$

Complete solution : $y = y_c + p_1 - p_2$

Thus $y = c_1 e^{2x} + c_2 e^{-6x} + \dfrac{x e^{2x}}{8} + \dfrac{3}{40}(\cos 2x + 2\sin 2x)$

---

64. *Solve :* $\dfrac{d^2y}{dx^2} - y = (1+x^2)e^x + x\sin x$

>> We have $(D^2-1)y = (1+x^2)e^x + x\sin x$

A.E is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$$\therefore \quad y_c = c_1 e^x + c_2 e^{-x}$$

$$y_p = \frac{(1+x^2)e^x}{D^2-1} + \frac{x\sin x}{D^2-1} = p_1 + p_2 \,(say)$$

$$p_1 = e^x \cdot \frac{1+x^2}{D^2-1} \quad \text{Now } D \to D+1$$

$$= e^x \cdot \frac{1+x^2}{(D+1)^2-1} = e^x \cdot \frac{1+x^2}{D^2+2D} = e^x \cdot \frac{x^2+1}{2D+D^2}$$

We need to employ division now.

$$\begin{array}{r|l l}
 & x^3/6 - x^2/4 + 3x/4 & \\
\hline
2D+D^2 & x^2+1 & \dfrac{x^2}{2D} = \displaystyle\int \frac{x^2}{2}\,dx = \frac{x^3}{6} \\
 & x^2+x & \\
\hline
 & -x+1 & \dfrac{-x}{2D} = \displaystyle\int \frac{-x}{2}\,dx = \frac{-x^2}{4} \\
 & -x-1/2 & \\
\hline
 & 3/2 & \dfrac{3/2}{2D} = \dfrac{3}{4}\displaystyle\int dx = \frac{3x}{4} \\
 & 3/2 & \\
\hline
 & 0 &
\end{array}$$

$$\therefore \quad \text{quotient} = \frac{x^3}{6} - \frac{x^2}{4} + \frac{3x}{4} = \frac{x}{12}(2x^2 - 3x + 9)$$

Hence $\quad p_1 = e^x \cdot \dfrac{x}{12}(2x^2 - 3x + 9)$

Next $\quad p_2 = \dfrac{x\sin x}{D^2-1}$ and we use $\dfrac{xV}{f(D)} = \left[x - \dfrac{f'(D)}{f(D)}\right]\dfrac{V}{f(D)}$

$$p_2 = \left[x - \frac{2D}{D^2-1}\right]\frac{\sin x}{D^2-1}$$

$$= \left[x - \frac{2D}{D^2-1}\right]\frac{\sin x}{-1-1} = \frac{-x\sin x}{2} + \frac{\cos x}{D^2-1} \quad ; \quad D^2 \to -1$$

$$p_2 = \frac{-x\sin x}{2} + \frac{\cos x}{-2} = \frac{-1}{2}\,(\,x\sin x + \cos x\,)$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2$

Thus $\boldsymbol{y = c_1 e^x + c_2 e^{-x} + \frac{x e^x}{12}(\,2x^2 - 3x + 9\,) - \frac{1}{2}(\,x\sin x + \cos x\,)}$

---

65. Solve : $(\,D^3 + D^2 - 4D - 4\,)\,y = 3e^{-x} - 4x - 6$

\>> A.E is $m^3 + m^2 - 4m - 4 = 0$

ie., $m^2(\,m+1\,) - 4(\,m+1\,) = 0$

or $(\,m+1\,)(\,m^2 - 4\,) = 0$

$\Rightarrow$ $m = -1,\ \pm 2$

$\therefore$ $y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-2x}$

$$y_p = \frac{3e^{-x}}{D^3 + D^2 - 4D - 4} - \frac{(\,4x+6\,)}{D^3 + D^2 - 4D - 4} = p_1 - p_2\ (\,say\,)$$

$$p_1 = \frac{3e^{-x}}{D^3 + D^2 - 4D - 4} = \frac{3e^{-x}}{-1+1+4-4} \quad (\,Dr. = 0\,)$$

$$p_1 = x\cdot\frac{3e^{-x}}{3D^2 + 2D - 4} = x\cdot\frac{3e^{-x}}{3-2-4} = -x\,e^{-x}$$

$$p_2 = \frac{4x+6}{-4-4D+D^2+D^3}$$ P.I is found by division.

$$\begin{array}{r|l} & -x-(\,1/2\,) \\ \hline -4-4D+D^2+D^3 & 4x+6 \\ & 4x+4 \\ \hline & 2 \\ & 2 \\ \hline & 0 \end{array} \qquad \therefore \quad p_2 = -x-(\,1/2\,)$$

Complete solution : $y = y_c + p_1 - p_2$

Thus $y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-2x} - x\,e^{-x} + x + (\,1/2\,)$

---

66. *Solve :* $\dfrac{d^2y}{dt^2} + 4y = t \sin t + \sin 2t$

>> We have $(D^2+4)\,y = t\sin t + \sin 2t$

A.E is $m^2+4 = 0 \Rightarrow m = \pm 2i$

$\therefore \quad y_c = c_1 \cos 2t + c_2 \sin 2t$

$$y_p = \frac{t\sin t + \sin 2t}{D^2+4} = \frac{t\sin t}{D^2+4} + \frac{\sin 2t}{D^2+4} = p_1 + p_2 \text{ (say)}$$

$$p_1 = \frac{t\sin t}{D^2+4} \text{ and let us use } \frac{tV}{f(D)} = \left[t - \frac{f'(D)}{f(D)}\right]\frac{V}{f(D)}$$

ie., $$= \left[t - \frac{2D}{D^2+4}\right]\frac{\sin t}{D^2+4} = \left[t - \frac{2D}{D^2+4}\right]\frac{\sin t}{-1^2+4}$$

ie., $$= \left[t - \frac{2D}{D^2+4}\right]\frac{\sin t}{3} = \frac{t\sin t}{3} - \frac{2\cos t}{3(D^2+4)}$$

ie., $$= \frac{t\sin t}{3} - \frac{2\cos t}{9} = \frac{1}{9}(3t\sin t - 2\cos t)$$

Also $p_2 = \dfrac{\sin 2t}{D^2+4}$ Now $D^2 \to -2^2$ and $Dr. = 0$

$$p_2 = t\,\frac{\sin 2t}{2D} = t\int \frac{\sin 2t}{2}\,dt = -\frac{t\cos 2t}{4}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2$

Thus $\boldsymbol{y = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{9}(3t\sin t - 2\cos t) - \frac{t\cos 2t}{4}}$

---

67. *Solve :* $\dfrac{d^2y}{dx^2} + 4y = x^2 + \cos 2x + 2^{-x}$

>> We have $(D^2+4)\,y = x^2 + \cos 2x + 2^{-x}$

A.E is given by $m^2+4 = 0$ and hence $m = \pm\, 2i$

$y_c = c_1 \cos 2x + c_2 \sin 2x$

$$y_p = \frac{x^2}{D^2+4} + \frac{\cos 2x}{D^2+4} + \frac{2^{-x}}{D^2+4}$$

$$y_p = p_1 + p_2 + p_3 \ (say)$$

$$p_1 = \frac{x^2}{D^2+4} \quad \text{P.I is found by division}$$

$$\begin{array}{r|l} & x^2/4 - 1/8 \\ \hline 4 + D^2 & x^2 \\ & \underline{x^2 + 1/2} \\ & -1/2 \\ & \underline{-1/2} \\ & 0 \end{array} \qquad \text{P.I} = \frac{x^2}{4} - \frac{1}{8} = \frac{1}{8}(2x^2 - 1)$$

$\log x^2$ [illegible]

$$p_1 = \frac{1}{8}(2x^2 - 1)$$

$$p_2 = \frac{\cos 2x}{D^2+4}$$

Replacing D^2 by $-2^2 = -4$, denominator becomes zero.

$$p_2 = x \cdot \frac{\cos 2x}{2D} = \frac{x \sin 2x}{4}$$

$$p_3 = \frac{2^{-x}}{D^2+4} = \frac{(e^{\log 2})^{-x}}{D^2+4} = \frac{e^{-\log 2 \cdot x}}{D^2+4} = \frac{e^{-\log 2 \cdot x}}{(-\log 2)^2+4}$$

$$p_3 = \frac{2^{-x}}{(\log 2)^2+4}$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2 + p_3$

Thus $\boldsymbol{y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(2x^2 - 1) + \frac{x \sin 2x}{4} + \frac{2^{-x}}{(\log 2)^2 + 4}}$

- -

68. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x} + \cos 2x + 4$

>> We have $(D^2 - 4D + 4)y = e^{2x} + \cos 2x + 4$

A.E is given by $m^2 - 4m + 4 = 0$

ie., $(m-2)^2 = 0 \Rightarrow m = 2, 2$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^{2x}$

$$y_p = \frac{e^{2x}}{D^2 - 4D + 4} + \frac{\cos 2x}{D^2 - 4D + 4} + \frac{4}{D^2 - 4D + 4}$$

$$= p_1 + p_2 + p_3 \ (say)$$

Now, $p_1 = \dfrac{e^{2x}}{D^2 - 4D + 4} = \dfrac{e^{2x}}{2^2 - 4(2) + 4} \quad (Dr. = 0)$

$$= x \cdot \frac{e^{2x}}{2D - 4} = x \cdot \frac{e^{2x}}{2(2) - 4} \qquad (Dr. = 0)$$

$$p_1 = x^2\, \frac{e^{2x}}{2}$$

$$p_2 = \frac{\cos 2x}{D^2 - 4D + 4} \quad \text{Replace } D^2 \text{ by } -4$$

$$p_2 = \frac{\cos 2x}{-4 - 4D + 4} = \frac{\cos 2x}{-4D} = -\frac{1}{4}\int \cos 2x\, dx = \frac{-\sin 2x}{8}$$

$$p_3 = \frac{4\, e^{0x}}{D^2 - 4D + 4} = \frac{4\, e^{0x}}{0 - 0 + 4} = \frac{4}{4} = 1$$

Complete solution : $y = y_c + y_p$ where $y_p = p_1 + p_2 + p_3$

Thus $\boldsymbol{y = (c_1 + c_2 x)\, e^{2x} + \dfrac{x^2 e^{2x}}{2} - \dfrac{\sin 2x}{8} + 1}$

---

69. Solve : $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 9y = 6e^3 x + 7e^{-2x} - \log 2$

>> We have $(D^2 - 6D + 9)\, y = 6e^3 x + 7e^{-2x} - \log 2$

A.E is given by $m^2 - 6m + 9 = 0$ or $(m - 3)^2 = 0 \Rightarrow m = 3, 3$

$\therefore \quad y_c = (c_1 + c_2 x)\, e^{3x}$

$$y_p = \frac{6e^3 x}{D^2 - 6D + 9} + \frac{7e^{-2x}}{D^2 - 6D + 9} - \frac{\log 2}{D^2 - 6D + 9}$$

$$= p_1 + p_2 + p_3 \ (say)$$